Dimensional Synthesis of Bennett Linkages

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- Introduction
- The Bennett linkage
- The workspace of the RR chain
- The cylindroid
- The design equations
- Bennett linkage coordinates
- Solving the design equations
- Examples
- Conclusions

Literature review

- ## Synthesis
  - Bennett, G.T., (1903) A New Mechanism. *Engineering*, vol. 76, pp. 777-778.
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- ## Analysis
  - Suh, C.H., and Radcliffe, C.W., (1978), *Kinematics and Mechanisms Design*. John Wiley.
  - Yu, H.C., (1981) The Bennett Linkage, its Associated Tetrahedron and the Hyperboloid of its Axes. *Mechanism and Machine Theory*, vol. 16, pp. 105-114.
  - Huang, C., (1996), The Cylindroid Associated with Finite Motions of the Bennett Mechanism. *Proc. of the ASME Design Engineering Technical Conferences*, Irvine, CA.
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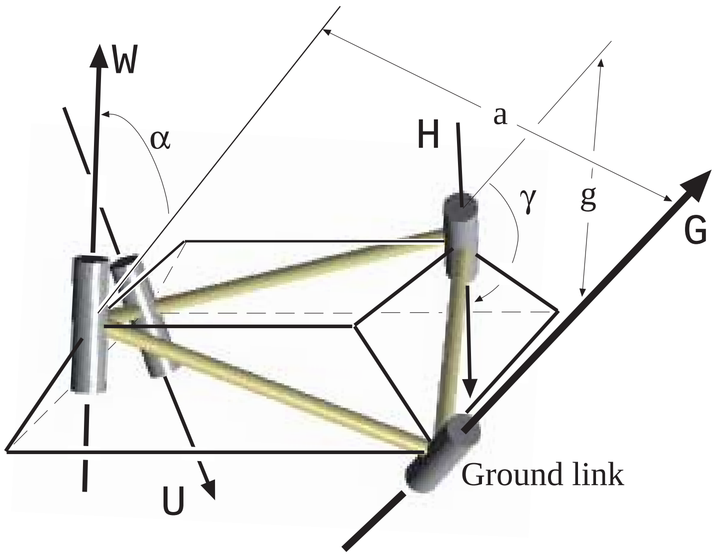
- ## Linear Systems of Screws
  - Hunt, K.H.,(1978), *Kinematic Geometry of Mechanisms*. Clarendon Press.
  - Parkin, I.A., (1997), Finding the Principal Axes of Screw Systems. *Proc. of the ASME Design Engineering Technical Conferences*, Sacramento, CA.

The Bennett Linkage - Spatial closed 4R chain



- Formed by connecting the end links of two spatial RR chains $\{G, W\}$ and $\{H, U\}$ to form a coupler link.
- **Mobility**: For a general 4R closed spatial chain:

$$M = 6(n-1) - \sum_{k=1}^{m} p_k c_k = 6.3 - 4.5 = -2$$

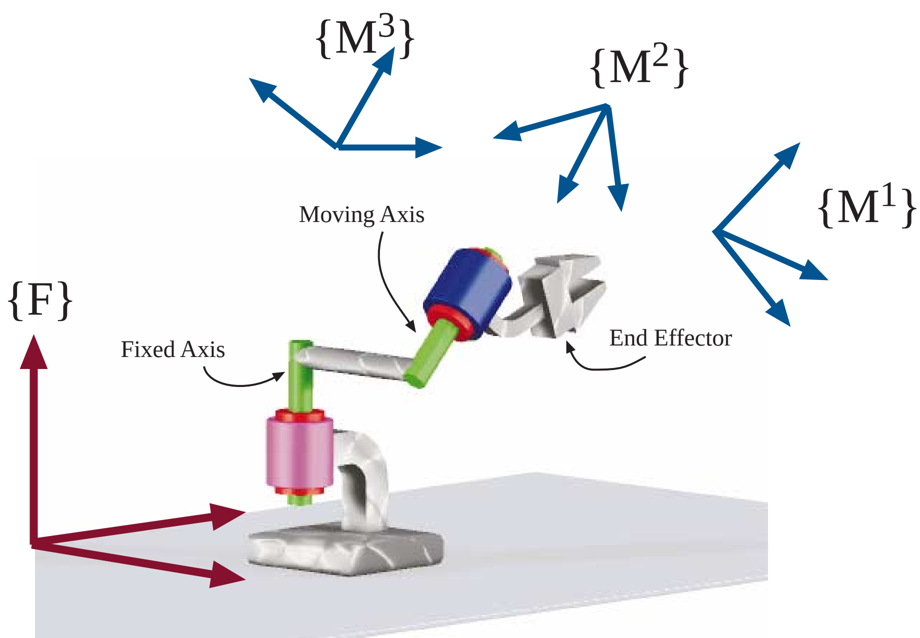
  However, the Bennett linkage can move with **one degree of freedom**.
- **Special geometry**:
  - Link lenght and twist angle (a, α) and (g, γ) must be the same for opposite sides.
  - Ratio of the sine of the twist angle over the link length:

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{g}$$

  - The joints of the Bennett linkage form the **vertices of a tetrahedron**

The Design Theory - The Workspace of the RR Chain

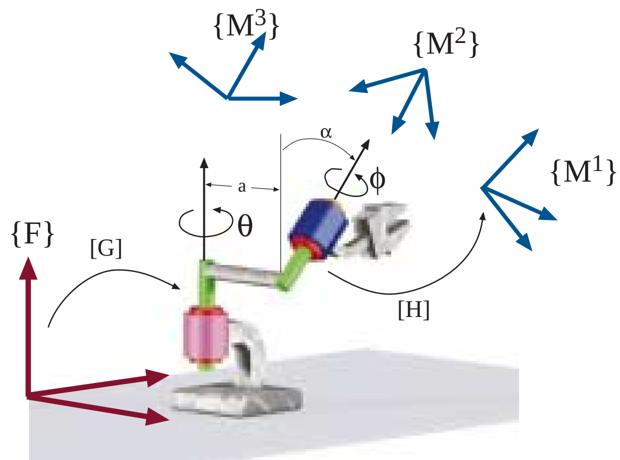
- **Synthesis Theory**- Find the kinematic chain that reaches exactly a number of specified positions



- The specified positions **must lie on the workspace of the chain**.

- The kinematics equation for the RR chain defines its workspace

The Kinematics Equation for the RR Chain



- The **kinematics equation in matrix form**:The set of displacements $[D(\theta, \phi)]$ of the RR chain.

$$[D] = [G][Z(\theta, 0)][X(\alpha, a)][Z(\phi, 0)][H]$$

- If we choose a reference configuration $[D_1]$, we can write the workspace of the **relative displacements** $[D_{1i}] = [D_i][D_1]^{-1}$.

$$[D_{1i}] = [T(\theta_i, \mathsf{G})][T(\phi_i, \mathsf{W})]$$

where

$[T(\Delta\theta, \mathsf{G})] = [G][Z(\theta, 0)][Z(\theta_0, 0)]^{-1}[G]^{-1}$,

$[T(\Delta\phi, \mathsf{W})] = ([G][[Z(\theta_0, 0)][X(\rho, r)])[Z(\phi, 0)][Z(\phi_0, 0)]^{-1}([G][[Z(\theta_0, 0)][X(\rho, r)])^{-1}$

The Kinematics Equation for the RR Chain

Dual quaternion formulation

- We can also formulate the workspace using **dual quaternions** to express the relative displacements.

The dual quaternion form of the workspace is given by:

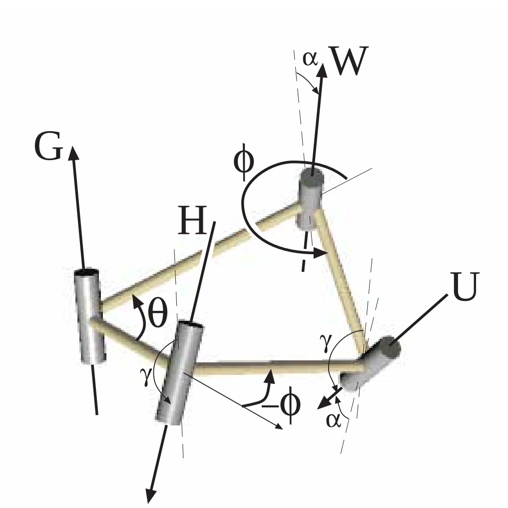
$$\widehat{D}_{1i} = \widehat{G}(\Delta\theta)\widehat{W}(\Delta\phi)$$

$$\cos(\frac{\widehat{\psi}_{1i}}{2}) = \cos\frac{\Delta\theta}{2}\cos\frac{\Delta\phi}{2} - \sin\frac{\Delta\theta}{2}\sin\frac{\Delta\phi}{2}\mathsf{G}\cdot\mathsf{W},$$
$$\sin(\frac{\widehat{\psi}_{1i}}{2})\mathsf{S}_{1i} = \sin\frac{\Delta\theta}{2}\cos\frac{\Delta\phi}{2}\mathsf{G} + \sin\frac{\Delta\phi}{2}\cos\frac{\Delta\theta}{2}\mathsf{W} + \sin\frac{\Delta\theta}{2}\sin\frac{\Delta\phi}{2}\mathsf{G}\times\mathsf{W}$$

where S_{1i} is the screw axis of the relative displacement and $\widehat{\psi}_{1i} = (\phi_{1i}, d_{1i})$ is the associated rotation about and slide along this axis for each displacement in the workspace.

- Every pair of values $\Delta\theta$ and $\Delta\phi$ defines a screw axis S_{1i} that represents a relative displacement from position 1 to position i.

The Workspace of the Bennett Linkage



- **Restriction to a Bennett linkage**: The angles θ and ϕ are not independent. There exist the input/coupler angular relation:

$$\tan\frac{\phi}{2} = -\frac{\sin\frac{\gamma+\alpha}{2}}{\sin\frac{\gamma-\alpha}{2}}\tan\frac{\theta}{2} = K\tan\frac{\theta}{2}$$

- **The workspace of the Bennett linkage**: The set of screw axes obtained applying the input/coupler relation to the workspace of the RR chain,

$$\tan(\frac{\widehat{\psi}_{1i}}{2})\mathsf{S}_{1i} = \frac{\mathsf{G} + K'\mathsf{W}^1 + K'\tan\frac{\theta}{2}\mathsf{G}\times\mathsf{W}^1}{\cot\frac{\theta}{2} - K'\tan\frac{\theta}{2}\mathsf{G}\cdot\mathsf{W}^1}.$$

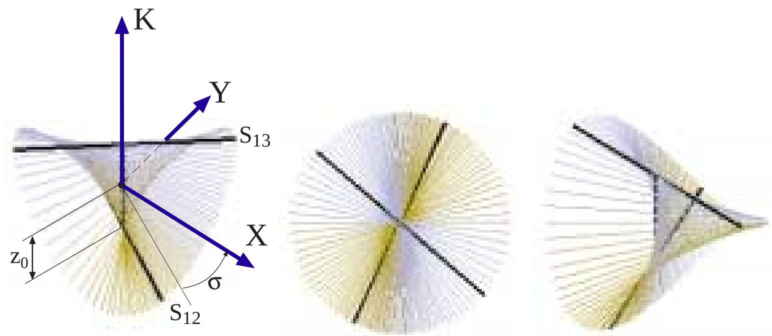
generates a **cylindroid**

The cylindroid

- Simply-Ruled surface that has a nodal line cutting all generators at right angles.

$$z(x^2 + y^2) + (P_X - P_Y)xy = 0$$

- It appears as generated by the real linear combination of two screws.
- The cylindroid has a set of **principal axes** located in the midpoint of the nodal line.

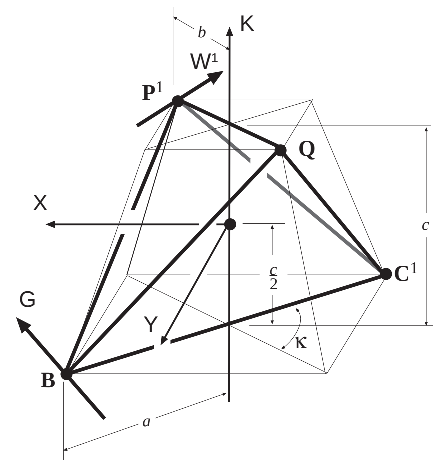


- The principal axes can be located from any pair of generators.

$$\tan 2\sigma = \frac{-(P_b - P_a)\cot\delta + d}{(P_b - P_a) + d\cot\delta}$$

$$z_0 = \frac{1}{2}(d - (P_b - P_a)\frac{\cos\delta}{\sin\delta})$$

Bennett linkage coordinates



- Yu, 1981: The Bennett linkage can be determined using a **tetrahedron** defined by four parameters $(a,\ b,\ c,\ \kappa)$.
- The **principal axes** are located in the middle of the tetrahedron.
- The joint axes $\mathbf{G}$ and $\mathbf{W}^1$ are given by the cross product of the edges. This ensures that the chosen points $\mathbf{B}$, $\mathbf{P}^1$ are on the common normal.

$$\mathsf{G} = K_g(\mathbf{Q}-\mathbf{B})\times(\mathbf{P}^1-\mathbf{B}) + \epsilon K_g \mathbf{B}\times\Big((\mathbf{Q}-\mathbf{B})\times(\mathbf{P}^1-\mathbf{B})\Big)$$

$$\mathsf{W}^1 = K_w(\mathbf{B}-\mathbf{P}^1)\times(\mathbf{C}^1-\mathbf{P}^1) + \epsilon K_w \mathbf{P}^1\times\Big((\mathbf{B}-\mathbf{P}^1)\times(\mathbf{C}^1-\mathbf{P}^1)\Big)$$

We obtain:

$$\mathsf{G} = K_g\left\{\begin{array}{c} 2bc\sin\frac{\kappa}{2} \\ 2bc\cos\frac{\kappa}{2} \\ 4ab\cos\frac{\kappa}{2}\sin\frac{\kappa}{2}\end{array}\right\} + \epsilon K_g\left\{\begin{array}{c} b\cos\frac{\kappa}{2}(4a^2\sin^2\frac{\kappa}{2}+c^2) \\ -b\sin\frac{\kappa}{2}(4a^2\cos^2\frac{\kappa}{2}+c^2) \\ 2abc(\cos^2\frac{\kappa}{2}-\sin^2\frac{\kappa}{2})\end{array}\right\}$$

and

$$\mathsf{W}^1 = K_w\left\{\begin{array}{c} -2ac\sin\frac{\kappa}{2} \\ 2ac\cos\frac{\kappa}{2} \\ 4ab\cos\frac{\kappa}{2}\sin\frac{\kappa}{2}\end{array}\right\} + \epsilon K_w\left\{\begin{array}{c} -a\cos\frac{\kappa}{2}(4b^2\sin^2\frac{\kappa}{2}+c^2) \\ -a\sin\frac{\kappa}{2}(4b^2\cos^2\frac{\kappa}{2}+c^2) \\ 2abc(\cos^2\frac{\kappa}{2}-\sin^2\frac{\kappa}{2})\end{array}\right\}$$

Using the principal axes and the tetrahedron formulation, we can write the coordinates of the joint axes of the Bennett linkage with only four parameters .

The design equations for an RR dyad

- **The constant dual angle constraint**: $\hat{\alpha} = (\alpha, a)$,the angle and distance between the fixed and moving axes,must remain constant during the movement.

$$\mathsf{G} \cdot [\hat{T}_{1i} - I]\mathsf{W}^1 = 0,\ i = 2, 3,$$

  Usign the equivalent screw triangle formulation and separating real and dual part,

  1. **The direction equations**

$$\tan \frac{\psi_{1i}}{2} = \frac{\mathbf{G} \cdot (\mathbf{S}_{1i} \times \mathbf{W}^1)}{(\mathbf{S}_{1i} \times \mathbf{G}) \cdot (\mathbf{S}_{1i} \times \mathbf{W}^1)}, \quad i = 2, 3.$$

  2. **The distance equations**

$$(\mathbf{B} - \mathbf{P}^1) \cdot \mathbf{S}_{1i} - \frac{t_{1i}}{2} = 0, \quad i = 2, 3.$$

- **The normal constraints**: The normal line to $\mathbf{G}$ and$\mathbf{W}$, $\mathbf{P}^i - \mathbf{B}$,remains the same.

$$\mathbf{G} \cdot ([T_{1i}]\mathbf{P}^1 - \mathbf{B}) = 0,$$
$$\mathbf{W}^1 \cdot (\mathbf{P}^1 - [T_{1i}]^{-1}\mathbf{B}) = 0, i = 1, 2, 3.$$

Total equations: $2(n-1) + 2n$
Total parameters: 10
Number of positions needed for a finite number of solutions: $n = 3$.

The standard algebraic formulation of the synthesis problem consists on solving ten equations in ten parameters .

Solving the design equations in the principal axes frame

- The six common normal constraints are automatically satisfied.
- We solve system of **four equations in four parameters** a**,** b**,** c**,** κ.

$$\tan\frac{\psi_{12}}{2} = \frac{\mathbf{G}\cdot(\mathbf{S}_{12}\times\mathbf{W}^1)}{(\mathbf{S}_{12}\times\mathbf{G})\cdot(\mathbf{S}_{12}\times\mathbf{W}^1)} \quad (1)$$

$$\tan\frac{\psi_{13}}{2} = \frac{\mathbf{G}\cdot(\mathbf{S}_{13}\times\mathbf{W}^1)}{(\mathbf{S}_{13}\times\mathbf{G})\cdot(\mathbf{S}_{13}\times\mathbf{W}^1)} \quad (2)$$

$$(\mathbf{B}-\mathbf{P}^1)\cdot\mathbf{S}_{12} - \frac{t_{12}}{2} = 0 \quad (3)$$

$$(\mathbf{B}-\mathbf{P}^1)\cdot\mathbf{S}_{13} - \frac{t_{13}}{2} = 0 \quad (4)$$

Solution for a and b :the distance equations (3) and (4) are linear in a, b.

$$\frac{t_{12}}{2} + (a-b)\cos\delta_1\cos\frac{\kappa}{2} + (a+b)\sin\delta_1\sin\frac{\kappa}{2} = 0$$

$$\frac{t_{13}}{2} + (a-b)\cos\delta_2\cos\frac{\kappa}{2} + (a+b)\sin\delta_2\sin\frac{\kappa}{2} = 0$$

Defining the constraints:

$$K_s = \frac{t_{12}\cos\delta_2 - t_{13}\cos\delta_1}{2\sin(\delta_1-\delta_2)}$$

$$K_d = \frac{t_{13}\sin\delta_1 - t_{12}\sin\delta_2}{2\sin(\delta_1-\delta_2)}$$

We obtain:

$$a = \frac{K_s}{2\sin\frac{\kappa}{2}} + \frac{K_d}{2\cos\frac{\kappa}{2}}$$

$$b = \frac{K_s}{2\sin\frac{\kappa}{2}} - \frac{K_d}{2\cos\frac{\kappa}{2}}.$$

Solution for c :

- Substitute the values of a and b in the direction equations (1) and (2) and make the algebraic substitution $y = \tan\frac{\kappa}{2}$.

$$\frac{\tan\frac{\psi_{12}}{2}(\frac{K_s^2}{K_d^2} - y^2) + c^2\frac{\tan\frac{\psi_{12}}{2}}{2K_d^2}(y^2(\cos 2\delta_1 - 1) + \cos 2\delta_1 + 1) - 2\frac{cy}{K_d}(\cos\delta_1 + \frac{K_s \sin\delta_1}{K_d})}{(\frac{K_s^2}{K_d^2} - y^2) + \frac{c^2}{2K_d^2}(y^2(\cos 2\delta_1 - 1) + \cos 2\delta_1 + 1)} = 0,$$

$$\frac{\tan\frac{\psi_{13}}{2}(\frac{K_s^2}{K_d^2} - y^2) + c^2\frac{\tan\frac{\psi_{13}}{2}}{2K_d^2}(y^2(\cos 2\delta_2 - 1) + \cos 2\delta_2 + 1) - 2\frac{cy}{K_d}(\cos\delta_2 + \frac{K_s \sin\delta_2}{K_d})}{(\frac{K_s^2}{K_d^2} - y^2) + \frac{c^2}{2K_d^2}(y^2(\cos 2\delta_2 - 1) + \cos 2\delta_2 + 1)} = 0,$$

- The numerator and denominator share the roots associated with $c = 0$, which are not a solution of the spatial problem. Eliminate them from the numerator forcing the linear system to have more solutions than the trivial.

$$\begin{bmatrix} \tan\frac{\psi_{12}}{2} & \frac{\tan\frac{\psi_{12}}{2}}{2K_d^2}(y^2 c(\cos 2\delta_1 - 1) + \cos 2\delta_1 + 1) - 2\frac{y}{K_d}(\cos\delta_1 + \frac{K_s \sin\delta_1}{K_d}) \\ \tan\frac{\psi_{13}}{2} & \frac{\tan\frac{\psi_{13}}{2}}{2K_d^2}(y^2 c(\cos 2\delta_2 - 1) + \cos 2\delta_2 + 1) - 2\frac{y}{K_d}(\cos\delta_2 + \frac{K_s \sin\delta_2}{K_d}) \end{bmatrix} \begin{Bmatrix} (K_s^2/K_d^2 - y^2) \\ c \end{Bmatrix} = 0.$$

- Making the determinant of the matrix equal to zero we obtain a linear equation in c.
- Define the constants

$$K_{12} = \frac{t_{12}/2}{\tan\frac{\psi_{12}}{2}}\left(\frac{1}{\sin^2\delta_1 - \sin^2\delta_2}\right)$$
$$K_{13} = \frac{t_{13}/2}{\tan\frac{\psi_{13}}{2}}\left(\frac{1}{\sin^2\delta_1 - \sin^2\delta_2}\right)$$

- We obtain the expression for c:

$$c = (K_{13} - K_{12})\sin\kappa$$

Solution for κ :

- Substitute the expressions for a, b, c in one of the direction equations, (1) or (2). We obtain a cubic polynomial in y^2.

$$P: \quad C_3 y^6 + C_2 y^4 + C_1 y^2 + C_0 = 0$$

- The coefficients are:

$$C_3 = -K_d^2,$$
$$C_2 = K_s^2 - 2K_d^2 + 4(K_{12} - K_{13})(K_{13}\sin^2\delta_1 - K_{12}\sin^2\delta_2),$$
$$C_1 = 2K_s^2 - K_d^2 - 4(K_{12} - K_{13})(K_{13}\cos^2\delta_1 - K_{12}\cos^2\delta_2),$$
$$C_0 = K_s^2.$$

- Solve the cubic polynomial for $z = y^2$. This polynomial has one and only one real positive root z_0:

$$P(0) = K_s^2$$
$$P(\infty) = -K_d^2$$
$$P(-1) = -4(K_{12} - K_{13})^2$$

- The square root of the positive root gives the two solutions for κ.

$$\tan\frac{\kappa}{2} = \pm\sqrt{z_0}$$

The Solutions

- The two sets of solutions $(a, b, c, +\kappa)$ and $(-b, -a, -c, -\kappa)$ correspond to both dyads of the Bennett mechanism:

Solution 1		Solution 2
G$(a,\ b,\ c,\ \kappa)$	$=$	H$(-b, -a, -c, -\kappa)$
W$(a,\ b,\ c,\ \kappa)$	$=$	U$(-b, -a, -c, -\kappa)$
H$(a,\ b,\ c,\ \kappa)$	$=$	G$(-b, -a, -c, -\kappa)$
U$(a,\ b,\ c,\ \kappa)$	$=$	W$(-b, -a, -c, -\kappa)$

- The synthesis procedure yields the two RR dyads that form a Bennett linkage.

Example

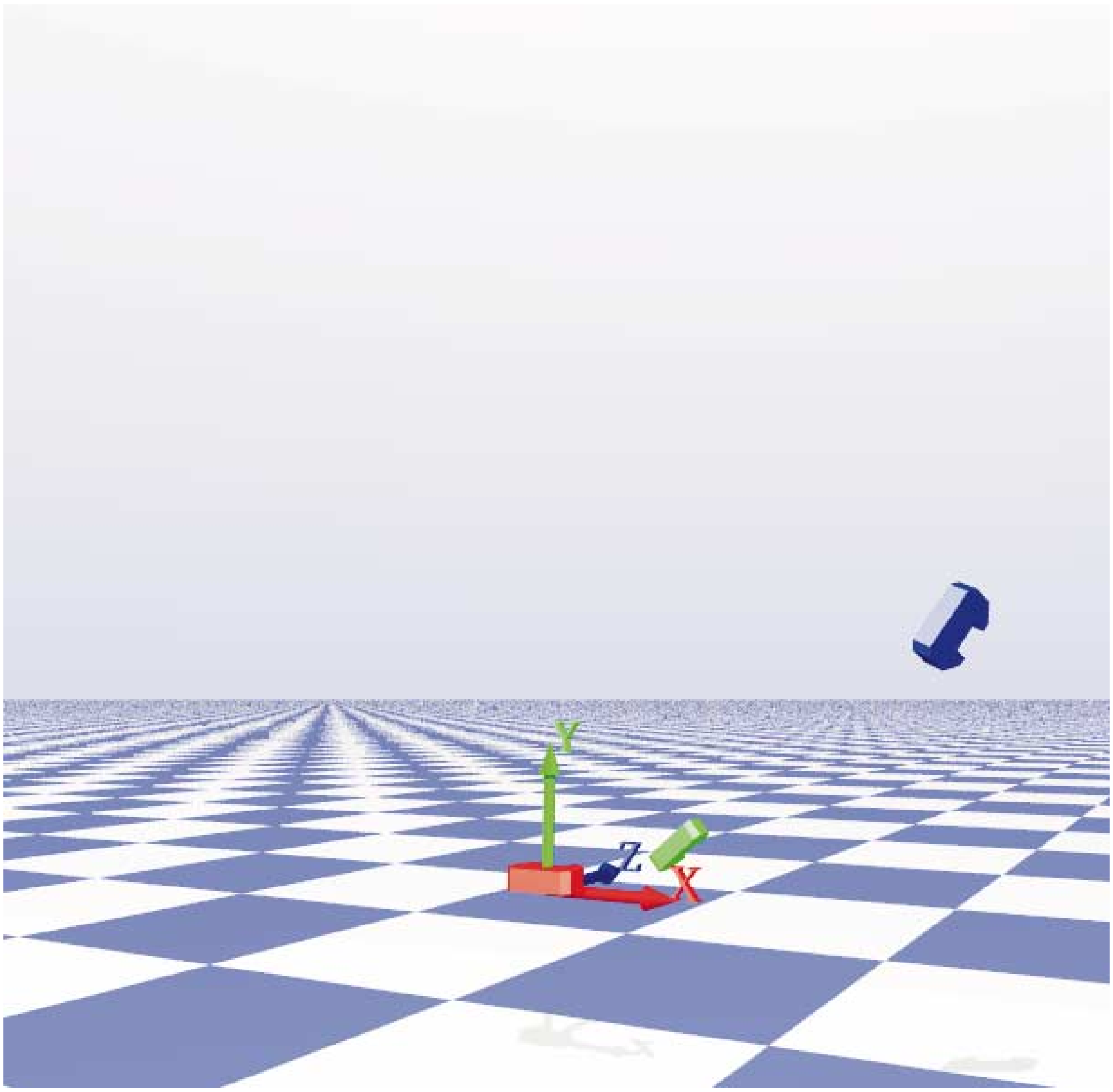
Tsai and Roth positions (Tsai and Roth, 1973)

- The specified positions:

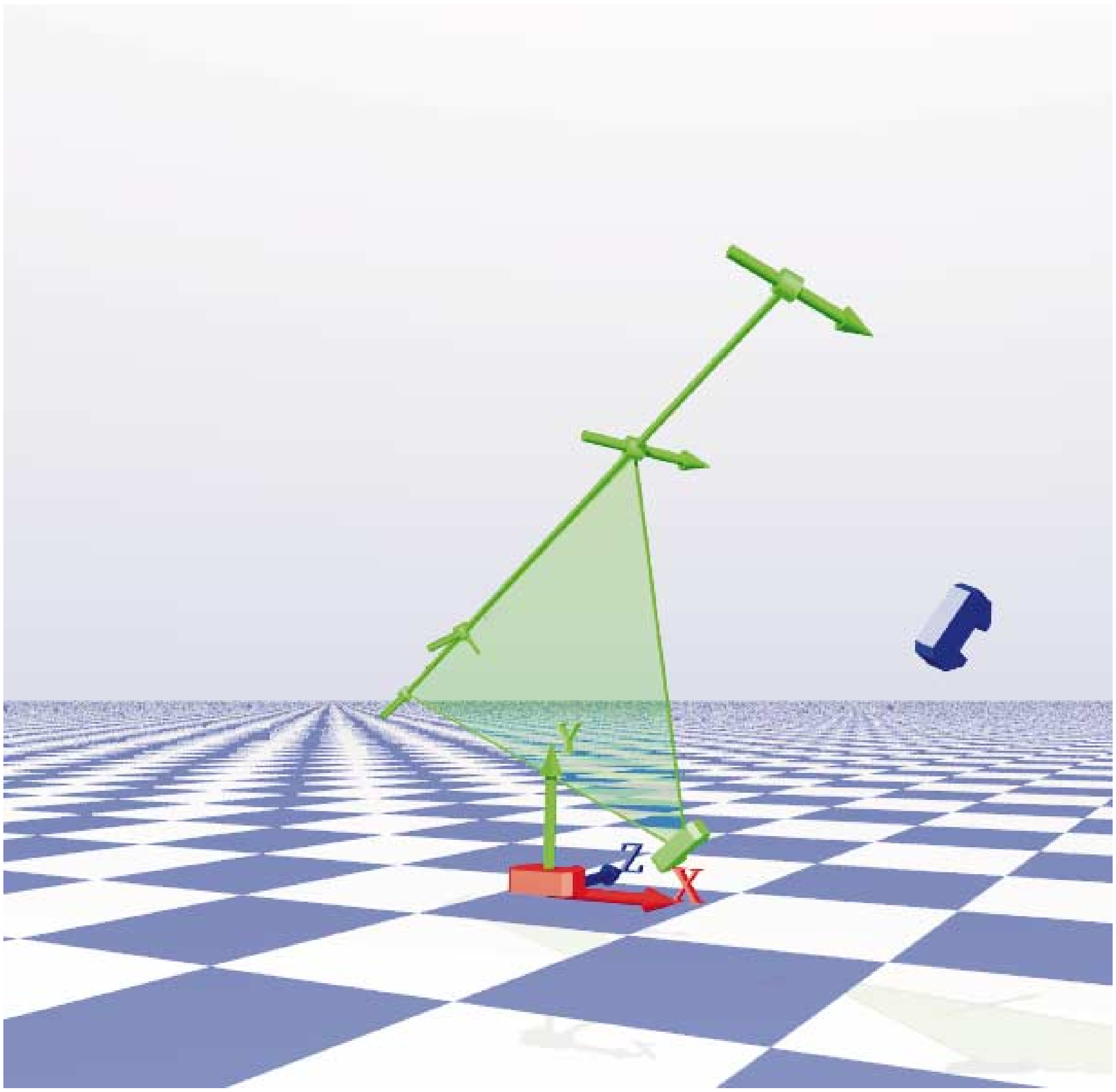
	x	y	z	θ	ϕ	ψ
M_1	0.0	0.0	0.0	0°	0°	0°
M_2	0.0	0.0	0.8	0°	0°	40°
M_3	1.11	0.66	0.05	18.8°	-28.0°	67.2°

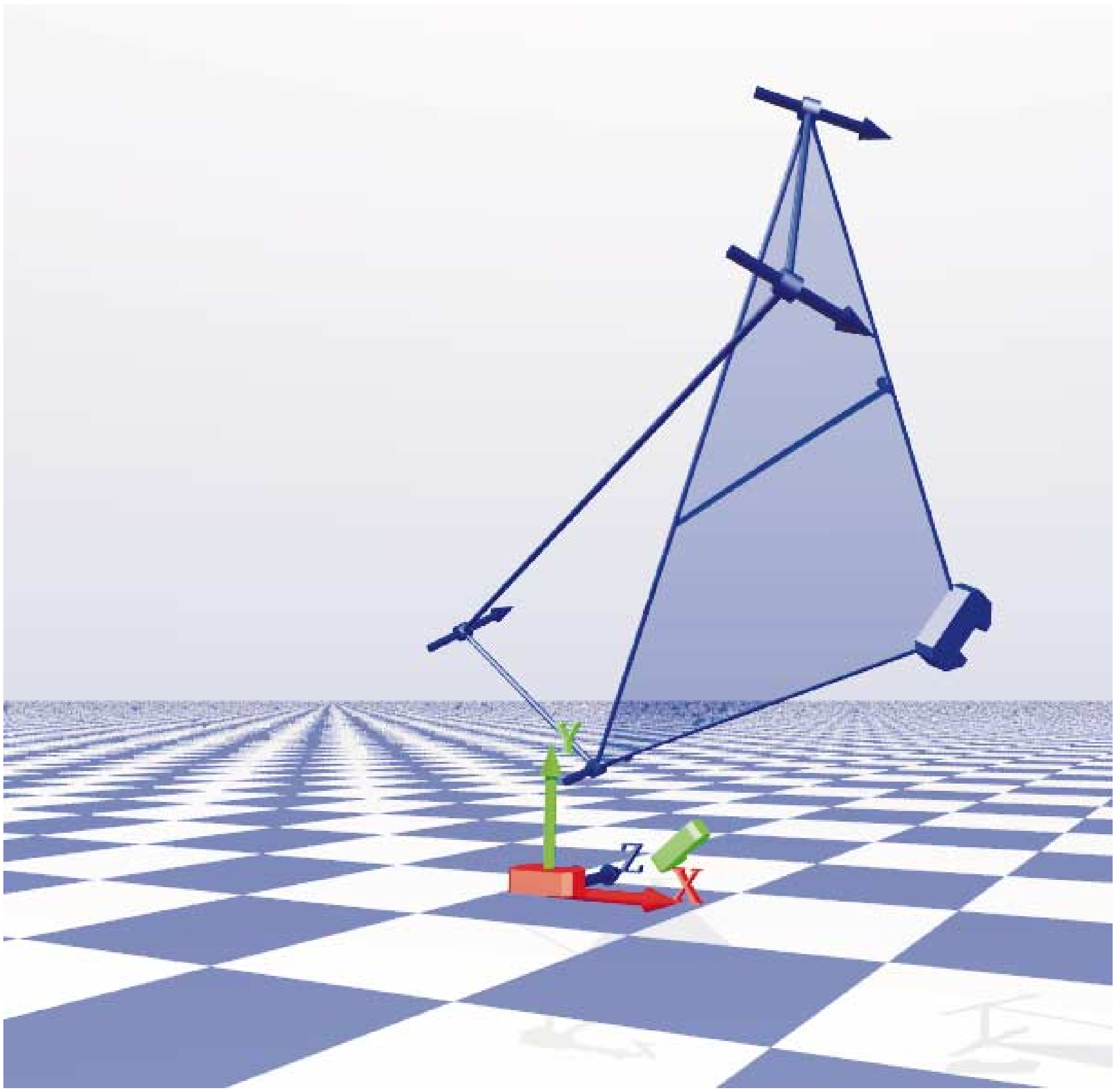
- The joint axes in the initial frame:

Axis	Line coordinates
G	$(0.36, 0.45, 0.81)\,,\ (0.26, 1.05, -0.70)$
W^1	$(0.60, 0.36, 0.72)\,,\ (0.87, 0.83, -1.14)$
H	$(0.60, -0.36, 0.72)\,,\ (0.87, -0.83, -1.14)$
U^1	$(0.36, -0.45, 0.81)\,,\ (0.26, -1.05, -0.70)$


Y
Z
X


Z
X


Y
Z
X


Y
Z
X

Conclusions

- Using the geometry of the RR chain to formulate the problem leads to a simple convenient set of equations.
- The design procedure for three positions for an RR chain yields a Bennett linkage.
- A *Mathematica* notebook with the complete synthesis procedure can be downloaded from: http://www.eng.uci.edu/ mccarthy/Pages/ResProjects.html
- The synthesis routine is to be used in a robot design environment for continuous tasks.