Clifford Algebra Exponentials and Planar Linkage Synthesis Equations

Alba Perez
e-mail: maperez@uci.edu

J. Michael McCarthy
e-mail: jmmccart@uci.edu

Robotics and Automation Laboratory,
University of California, Irvine, Irvine, CA 92697

*This paper uses the exponential defined on a Clifford algebra of planar projective space to show that the "standard-form" design equations used for planar linkage synthesis are obtained directly from the relative kinematics equations of the chain. The relative kinematics equations of a serial chain appear in the matrix exponential formulation of the kinematics equations for a robot. We show that formulating these same equations using a Clifford algebra yields design equations that include the joint variables in a way that is convenient for algebraic manipulation. The result is a single formulation that yields the design equations for planar 2R dyads, 3R triads, and n*R *single degree-of-freedom coupled serial chains and facilitates the algebraic solution of these equations including the inverse kinematics of the chain. These results link the basic equations of planar linkage design to standard techniques in robotics.* [DOI: 10.1115/1.1904047]

1 Introduction

A planar serial chain consists of a sequence of rigid links connected by revolute (hinged) or prismatic (sliding) joints such that the links of the chain are constrained to move parallel to a reference plane. This means the axes of the revolute joints are each perpendicular to this plane, and the sliding directions of the prismatic joints are parallel to it. The kinematics equations of these chains are obtained using the standard techniques of robotics, see for example Craig [1] or Tsai [2]. In this paper, we show that these equations can be used to formulate design equations for the chain that are equivalent to the "standard-form equations" used by Sandor and Erdman [3]. The benefit of this approach is that it can be generalized to formulate design equations for spatial serial chains.

The standard-form equation is the vector closure equation that defines the relative positions of a planar 2R chain (Fig. 1), called a *dyad*—R denotes a revolute, or hinged, joint. Lin and Erdman [4] generalized this approach to obtain design equations for planar 3R and 4R chains called triads and quadriads. Chase et al. [5] applied triad synthesis to the design of planar six-bar linkages, and Subbian and Flugrad [6] used it to design an eight-bar linkage. Krovi et al. [7] further extended the standard-form equation to design nR planar serial chains in which the n joints are connected by a cable drive. The result is a "single degree-of-freedom coupled serial chain" that they use to design an assistive device.

For related work, see Balli and Chand [8] who use these standard form equations to design adjustable planar five-bar linkages, and Holte et al. [9], who formulate synthesis equations for planar 2R chains that allows imprecision in the specification of the task positions.

This paper implements the approach to robot synthesis introduced by Mavroidis et al. [10] and Lee and Mavroidis [11] and expanded by Perez [12]. The Clifford algebra formulation of the kinematics equations links these results to the technique of kinematic mappings used by Bottema and Roth [13] and DeSa and Roth [14] to study and classify planar motion. Ravani and Roth [15], Larochelle and McCarthy [16], and Murray et al. [17] use this mapping to design planar linkages and manipulator systems.

Our contribution is a formulation that can be used to derive synthesis equations for any planar serial chain, and that can be generalized for the design of spatial serial chains [26]. We demonstrate the use of this theory with the synthesis of 2R and 3R planar chains.

2 The Kinematics Equations for Serial Chains

We define the kinematics equations of a general serial chain using the Denavit-Hartenberg convention in robotics and transform it to a "product of exponentials" formulation using relative transformations. Much of this is well-known; however, we develop it in detail to lay the foundation for a different interpretation of the kinematics equations.

Let S_i, $i=1,\ldots n$, denote the n joint axes in a spatial serial chain—we will limit these axes to be perpendicular to a reference plane when we specialize to planar chains. Introduce the line $\mathsf{A}_{i,i+1}$ which is the common normal to the axes S_i and S_{i+1}. See Fig. 2. We now set the origin of the joint frame T_i at the intersection of S_i and $\mathsf{A}_{i,i+1}$, such that the z axis is S_i and the x axis is $\mathsf{A}_{i,i+1}$, see McCarthy [18].

This allows us to write the kinematics equations of the chain in the form

$$[D]=[G][Z(\theta_1,d_1)][X(\alpha_{12},a_{12})][Z(\theta_2,d_2)]\ldots[X(\alpha_{n-1,n},a_{n-1,n})]\times[Z(\theta_n,d_n)][H] \tag{1}$$

where $[Z(\theta_i,d_i)]$ and $[X(\alpha_{i,i+1},a_{i,i+1})]$ are the 4×4 homogeneous transforms

$$[Z(\theta_i,d_i)]=\begin{bmatrix}\cos\theta_i & -\sin\theta_i & 0 & 0\\ \sin\theta_i & \cos\theta_i & 0 & 0\\ 0 & 0 & 1 & d_i\\ 0 & 0 & 0 & 1\end{bmatrix} \text{ and}$$

$$[X(\alpha_{i,i+1},a_{i,i+1})]=\begin{bmatrix}1 & 0 & 0 & a_{i,i+1}\\ 0 & \cos\alpha_{i,i+1} & -\sin\alpha_{i,i+1} & 0\\ 0 & \sin\alpha_{i,i+1} & \cos\alpha_{i,i+1} & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{2}$$

The parameters θ_i and d_i define the rotation of a revolute joint and slide of a prismatic joint, respectively. The parameters $\alpha_{i,i+1}$ and $a_{i,i+1}$ are the dimensional parameters that define the size of each link in the chain. The transformation $[G]$ locates the base of the robot in the world frame, and $[H]$ locates the tool frame relative to the last joint frame.

Contributed by Mechanisms and Robotics Committee for publication in the JOURNAL OF MECHANICAL DESIGN. Manuscript received June 24, 2004; Final manuscript received November 1, 2004. Review conducted by: G. R. Pennock.

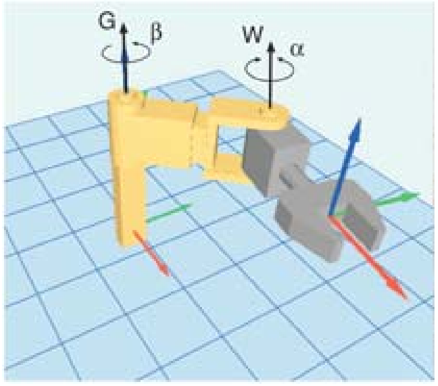

Fig. 1 A planar 2R serial chain

In what follows, we use the kinematics equations and a given set of task positions to determine the dimensional parameters of the chain. To do this it is more convenient to use the coordinates of joint axes, rather rather than the parameters $\alpha_{i,i+1}$ and $a_{i,i+1}$, as design variables. For this reason, we formulate the kinematics equations in terms of matrix exponentials.

2.1 Product of Exponentials. The product of exponentials formulation of the kinematics equations defines the 4×4 homogeneous transform directly in terms of the Plucker coordinates of the screw axis of the transformation. In particular, consider the displacement $[T(\phi,k,\mathsf{S})]$ with screw axis $\mathsf{S}=(\mathbf{S},\mathbf{C}\times\mathbf{S})$ defined such that it rotates the angle ϕ around S and slides the distance k along this axis. This displacement can be defined as the matrix exponential of the twist matrix formed by these parameters.

Let $\lambda=k/\phi$ be the pitch of the displacement $[T(\phi,k,\mathsf{S})]$, then we have the screw $\mathsf{J}=(\mathbf{S},\mathbf{V})=(\mathbf{S},\mathbf{C}\times\mathbf{S}+\lambda\mathbf{S})$, which defines the 4×4 twist matrix

$$J=\begin{bmatrix} 0 & -s_z & s_y & v_x \\ s_z & 0 & -s_x & v_y \\ -s_y & s_x & 0 & v_z \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

The 4×4 homogeneous transform $[T(\phi,k,\mathsf{S})]$ is now given by the matrix exponential

$$[T(\phi,k,\mathsf{S})]=e^{\phi J} \tag{4}$$

see Murray et al. [19].

The matrix exponentials for $[Z(\theta_i,d_i)]$ and $[X(\alpha_{i,i+1},a_{i,i+1})]$ have a particularly simple form. The screws defined for these two transformations are $\mathsf{K}=(\vec{k},\mu\vec{k})$ and $\mathsf{I}=(\vec{i},\nu\vec{i})$, where $\mu=d_i/\theta_i$ and $\nu=a_{i,i+1}/\alpha_{i,i+1}$ are their respective pitches. Thus, we have

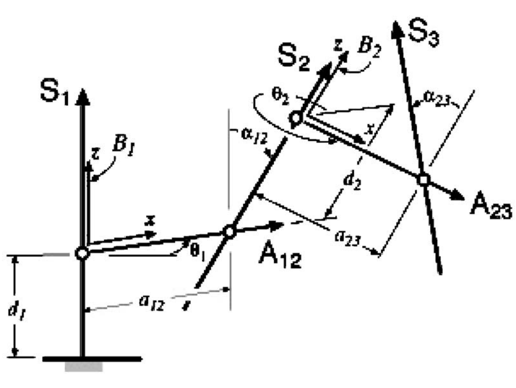

Fig. 2 Local frames for a serial chain

$$[Z(\theta_i,d_i)]=e^{\theta_i K} \quad \text{and} \quad [X(\alpha_{i,i+1},a_{i,i+1})]=e^{\alpha_{i,i+1}I} \tag{5}$$

and the kinematics Eqs. (1) become

$$[D]=[G]e^{\theta_1 K}e^{\alpha_{12}I}e^{\theta_2 K}\ldots e^{\alpha_{n-1,n}I}e^{\theta_n K}[H] \tag{6}$$

This is termed the *product of exponentials* form of the kinematics equations.

2.2 Relative Displacements. A useful form of the kinematics equations is obtained by choosing a reference configuration of the chain, $[D_0]$, defined by the joint angle vector $\vec{\theta}_0$, and determining the relative displacements $[D(\Delta\vec{\theta})]=[D(\vec{\theta})][D(\vec{\theta}_0)]^{-1}$. This form of the relative transformation operates on coordinates in the world frame. The transformation $[D_0]$ is usually selected to be the configuration in which the joint parameters are zero, called the *zero reference position* by Gupta [20].

For our purposes, the reference configuration $[D_0]$ can be any position of the end-effector specified by the vector of joint parameters $\vec{\theta}_0$, such that movement to a new configuration is defined by the joint parameter vector $\Delta\vec{\theta}=\vec{\theta}-\vec{\theta}_0$. From Eq. (1), we compute the relative kinematics equations

$$\begin{aligned}[D(\Delta\vec{\theta})]&=[D(\vec{\theta})][D(\vec{\theta}_0)]^{-1}=([G][Z(\theta_1,d_1)]\ldots[Z(\theta_n,d_n)][H])\\&\quad\times([G][Z(\theta_{10},d_{10})]\ldots[Z(\theta_{n0},d_{n0})][H])^{-1}\end{aligned} \tag{7}$$

In order to expand this equation, we introduce the partial displacements

$$[A_{i0}]=[G][Z(\theta_{10},d_{10})][X(\alpha_{12},a_{12})]\ldots[X(\alpha_{i-1,i},a_{i-1,i})] \tag{8}$$

where, for example

$$[A_{10}]=[G], \quad \text{and} \quad [A_{20}]=[G][Z(\theta_{10},d_{10})][X(\alpha_{12},a_{12})].$$

Now, insert the identity $[Z(\theta_{i,0})]^{-1}[A_{i0}]^{-1}[A_{i0}][Z(\theta_{i0})]=[I]$ after the first $n-1$ joint transforms $[Z(\theta_i,d_i)]$ in Eq. (7) in order to obtain the sequence of terms

$$\begin{aligned}[T(\Delta\theta_i,\mathsf{S}_i)]&=[A_{i0}][Z(\theta_i,d_i)][Z(\theta_{i,0})]^{-1}[A_{i0}]^{-1}\\&=[A_{i0}][Z(\Delta\theta_i,\Delta d_i)][A_{i0}]^{-1}.\end{aligned} \tag{9}$$

The result is that the relative kinematics equations take the form

$$[D(\Delta\vec{\theta})]=[T(\Delta\theta_1,\mathsf{S}_1)][T(\Delta\theta_2,\mathsf{S}_2)]\ldots[T(\Delta\theta_n,\mathsf{S}_n)] \tag{10}$$

where S_i are the Plucker coordinates of each joint axis obtained by transforming the joint screw K to the world frame by the coordinate transformations defined in (9).

Using the exponential form of the transformations $[T(\Delta\theta_i,\mathsf{S}_i)]$, we obtain the relative kinematics Eqs. (10) as

$$[D(\Delta\vec{\theta})]=e^{\Delta\theta_1 S_1}e^{\Delta\theta_2 S_2}\ldots e^{\Delta\theta_n S_n} \tag{11}$$

where

$$S_i=A_{i0}KA_{i0}^{-1} \tag{12}$$

The product of exponentials form of the kinematics Eqs. (6) becomes

$$[D]=[D(\Delta\vec{\theta})][D_0]=e^{\Delta\theta_1 S_1}e^{\Delta\theta_2 S_2}\ldots e^{\Delta\theta_n S_n}[D_0] \tag{13}$$

Notice that in this equation the coordinates of the joint axes are defined in the world frame such that the chain aligns the end-effector with the reference position.

3 Planar Serial Chains

We now specialize the kinematics equations defined earlier to the case of planar serial chains. It is convenient for our purposes to focus on chains consisting only of revolute joints, the nR chain.

The Plucker coordinates of the axis of a typical revolute joint in a planar chain are given by $\mathsf{J}=(\vec{k},\mathbf{C}\times\vec{k})$, where $\vec{k}=(0,0,1)$ is

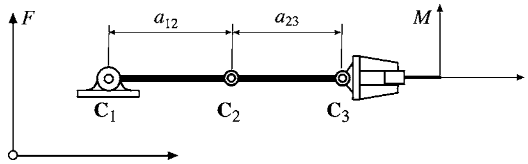

Fig. 3 A planar 3R chain in the reference configuration

directed along the z axis of the base frame, and $\mathbf{C}=(c_x,c_y,0)$ is the point of intersection of this axis with the x-y plane. The associated twist matrix J is

$$J=\begin{bmatrix} 0 & -1 & 0 & -c_y \\ 1 & 0 & 0 & c_x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{14}$$

Let the transformation to the base of the chain be a translation by the vector $\mathbf{G}=(g_x,g_y,0)$, then the zero configuration of the nR planar chain has the points $\mathbf{C}_i$, $i=1,\ldots,n$ on the joint axes J_i distributed along a line parallel to x axis (see Fig. 3), such that

$$\mathbf{C}_1=\begin{Bmatrix} g_x \\ g_y \\ 0 \end{Bmatrix},\quad \mathbf{C}_2=\begin{Bmatrix} g_x+a_{12} \\ g_y \\ 0 \end{Bmatrix},\ldots,$$
$$\mathbf{C}_n=\begin{Bmatrix} g_x+a_{12}+a_{23}+\cdots+a_{n-1,n} \\ g_y \\ 0 \end{Bmatrix} \tag{15}$$

Substituting these points into (14) we obtain a twist matrix J_i for each revolute joint, and the product of exponentials kinematics equations

$$[D(\vec{\theta})]=e^{\Delta\theta_1 J_1}e^{\Delta\theta_2 J_2}\ldots e^{\Delta\theta_n J_n}[D_0] \tag{16}$$

The zero frame transformation $[D_0]$ can be defined by introducing $[C]$ which is the translation by the vector $\mathbf{c}=(a_{12}+a_{23}+\cdots+a_{n-1,n})\vec{i}$ along the chain in the zero configuration, so we have

$$[D_0]=[G][C][H] \tag{17}$$

The matrix exponential defining the rotation about $\mathbf{J}$ by the angle $\Delta\theta$ can be computed using formulas in Murray et al. [19] to yield

$$e^{\Delta\theta J}=\begin{bmatrix} \cos\Delta\theta & -\sin\Delta\theta & 0 & (1-\cos\Delta\theta)c_x+\sin\Delta\theta c_y \\ \sin\Delta\theta & \cos\Delta\theta & 0 & -\sin\Delta\theta c_x+(1-\cos\Delta\theta)c_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{18}$$

This matrix defines a displacement consisting of a planar rotation about the point $\mathbf{C}$, called the *pole of the displacement*.

3.1 Complex Number Kinematics Equations. It is convenient at this point to introduce the complex numbers $e^{i\Delta\theta}=\cos\Delta\theta+i\sin\Delta\theta$ and $\mathbf{C}=c_x+ic_y$ to simplify the representation of the displacement (18).

Let $\mathbf{X}_1=x+iy$ be the coordinates of a point in the world frame in the first position and $\mathbf{X}_2=X+iY$ be its coordinates in the second position, then this transformation becomes

$$\mathbf{X}_2=e^{i\Delta\theta}\mathbf{X}_1+(1-e^{i\Delta\theta})\mathbf{C} \tag{19}$$

The complex numbers $e^{i\Delta\theta}$ and $(1-e^{i\Delta\theta})\mathbf{C}$ define the rotation and translation that form the planar displacement $e^{\Delta\theta J}$. The point $\mathbf{C}$ is the pole of the displacement, and the translation vector $\mathbf{D}$ associated with this displacement is given by

$$\mathbf{D}=(1-e^{i\Delta\theta})\mathbf{C} \tag{20}$$

The composition of the exponentials $e^{\theta_1 C_1}$ and $e^{\theta_2 C_2}$ that define rotations about the points $\mathbf{C}_1$ and $\mathbf{C}_2$, respectively, yields

$$e^{\phi P}=e^{\theta_1 C_1}e^{\theta_2 C_2}\quad \text{or}$$
$$\begin{aligned}[e^{i\phi},(1-e^{i\phi})\mathbf{P}]&=[e^{i\theta_1},(1-e^{i\theta_1})\mathbf{C}_1][e^{i\theta_2},(1-e^{i\theta_2})\mathbf{C}_2]\\&=[e^{i(\theta_1+\theta_2)},(1-e^{i\theta_1})\mathbf{C}_1+e^{i\theta_1}(1-e^{i\theta_2})\mathbf{C}_2]\end{aligned} \tag{21}$$

Here $\mathbf{P}$ denotes the pole of the composite displacement.

The complex form of the relative kinematics Eqs. (11) is seen to be

$$[D(\Delta\vec{\theta})]=[e^{i\Delta\theta_1},(1-e^{i\Delta\theta_1})\mathbf{C}_1][e^{i\Delta\theta_2},(1-e^{i\Delta\theta_2})\mathbf{C}_2]\ldots[e^{i\Delta\theta_n},(1-e^{i\Delta\theta_n})\mathbf{C}_n] \tag{22}$$

If we define the relative displacement of the end-effector to be $[D]=[e^{i\Delta\phi},(1-e^{i\Delta\phi})\mathbf{P}]$, then we can expand this equation and equate the rotation and translation components to obtain

$$e^{i\Delta\phi}=e^{i\Delta\theta_1}e^{i\Delta\theta_2}\ldots e^{i\Delta\theta_n}=e^{i(\Delta\theta_1+\Delta\theta_2+\ldots+\Delta\theta_n)}$$
$$\begin{aligned}(1-e^{i\Delta\phi})\mathbf{P}&=(1-e^{i\Delta\theta_1})\mathbf{C}_1+e^{i\Delta\theta_1}(1-e^{i\Delta\theta_2})\mathbf{C}_2+\ldots\\&\quad+e^{i(\Delta\theta_1+\Delta\theta_2+\ldots+\Delta\theta_{n-1})}(1-e^{i\Delta\theta_n})\mathbf{C}_n\end{aligned} \tag{23}$$

These complex vector equations can be used to design planar nR serial chains. We will see shortly that they are exactly Sandor and Erdman's standard form equations. However, in the next section we introduce an equivalent set of design equations using the Clifford algebra form of the kinematics equations.

4 The Even Clifford Algebra $C^+(P^2)$

The even Clifford algebra of the projective plane P^2 can be viewed as a generalization of complex numbers. It is a vector space with a product operation that is linked to a scalar product. This Clifford algebra has an even subalgebra, $C^+(P^2)$, which is a set of four dimensional elements of the form

$$A=a_1 i\epsilon+a_2 j\epsilon+a_3 k+a_4 \tag{24}$$

The basis elements $i\epsilon$, $j\epsilon$, k, and 1 satisfy the following multiplication table:

	$i\epsilon$	$j\epsilon$	k	1
$i\epsilon$	0	0	$-j\epsilon$	$i\epsilon$
$j\epsilon$	0	0	$i\epsilon$	$j\epsilon$
k	$j\epsilon$	$-i\epsilon$	-1	k
1	$i\epsilon$	$j\epsilon$	k	1

(25)

These basis elements are well-known components of the hypercomplex number known as a dual quaternion, McCarthy [18].

A point in the plane P^2, $\mathbf{C}=(c_x,c_y,1)$, can be identified with the Clifford algebra element

$$\mathsf{C}=k-(c_x+kc_y)j\epsilon=k-\mathbf{C}j\epsilon, \tag{26}$$

where $\mathbf{C}=c_x+kc_y$. The product operation of the Clifford algebra allows us to compute the exponential of C, given by

$$e^{\theta\mathsf{C}}=1+\theta\mathsf{C}+\frac{\theta^2}{2}\mathsf{C}^2+\frac{\theta^3}{3!}\mathsf{C}^3+\cdots \tag{27}$$

Using the identities

$$\mathsf{C}^2=-1,\quad \mathsf{C}^3=-\mathsf{C},\quad \mathsf{C}^4=1,\cdots \tag{28}$$

Equation (27) simplifies to become

$$e^{\theta\mathsf{C}}=\cos\theta+\sin\theta\mathsf{C}. \tag{29}$$

Introduce the complex exponential $e^{k\theta}=\cos\theta+k\sin\theta$ (recall $k^2=-1$) into this equation, so that we have

$$e^{\theta\mathbf{C}} = \left(1 + \tfrac{1}{2}(1 - e^{k2\theta})\mathbf{C}i\epsilon\right)e^{k\theta}. \tag{30}$$

McCarthy [21] shows that (30) defines the planar displacement consisting of a rotation by $\phi = 2\theta$ about the point $\mathbf{C}$, so we have

$$e^{(\phi/2)\mathbf{C}} = \left(1 + \tfrac{1}{2}(1 - e^{k\phi})\mathbf{C}i\epsilon\right)e^{k\phi/2} = \tfrac{1}{2}(e^{-k\phi/2} - e^{k\phi/2})\mathbf{C}i\epsilon + e^{k\phi/2}$$
$$= c_y \sin\frac{\phi}{2}i\epsilon - c_x \sin\frac{\phi}{2}j\epsilon + \sin\frac{\phi}{2}k + \cos\frac{\phi}{2} \tag{31}$$

Bottema and Roth [13] used the components of this Clifford algebra element as a kinematic mapping to study planar displacements. DeSa and Roth [14] use this mapping to classify planar motion, and Ravani and Roth [15] use it to design planar linkages.

Notice that the translation vector of this displacement is $\mathbf{d} = (1 - e^{k\phi})\mathbf{C}$. Thus, we can also write (31) in the form

$$e^{(\phi/2)\mathbf{C}} = \left(1 + \tfrac{1}{2}\mathbf{d}i\epsilon\right)e^{k\phi/2} \tag{32}$$

Clifford Algebra Kinematics Equations. The relative kinematics equations of an nR planar chain (22) can be written in terms of the Clifford algebra elements (31) to define

$$\begin{aligned}-\sin\tfrac{\Delta\phi}{2}\mathbf{P}j\epsilon + e^{k\Delta\phi/2} &= \left(-\sin\tfrac{\Delta\theta_{1j}}{2}\mathbf{C}_1 j\epsilon + e^{k\Delta\theta_1/2}\right)\left(-\sin\tfrac{\Delta\theta_2}{2}\mathbf{C}_2 j\epsilon\right. \\ &\left.+ e^{k\Delta\theta_2/2}\right)\cdots\left(-\sin\tfrac{\Delta\theta_n}{2}\mathbf{C}_n j\epsilon + e^{k\Delta\theta_n/2}\right)\end{aligned} \tag{33}$$

Expand this equations and equate coefficients of the basis elements to obtain

$$e^{k\Delta\phi/2} = e^{k(\Delta\theta_1 + \Delta\theta_2 + \cdots + \Delta\theta_n)/2},$$

$$\begin{aligned}\sin\tfrac{\Delta\phi}{2}\mathbf{P} &= \sin\tfrac{\Delta\theta_1}{2}\mathbf{C}_1 e^{-k(\Delta\theta_2 + \ldots + \Delta\theta_n)/2} \\ &+ e^{k\Delta\theta_1/2}\sin\tfrac{\Delta\theta_2}{2}\mathbf{C}_2 e^{-k(\Delta\theta_3 + \ldots + \Delta\theta_n)/2} + \ldots \\ &+ e^{k(\Delta\theta_1 + \Delta\theta_2 + \ldots + \Delta\theta_{n-1})/2}\sin\tfrac{\Delta\theta_n}{2}\mathbf{C}_n\end{aligned} \tag{34}$$

These equations are equivalent to the complex vector equations presented earlier. In fact, multiplication of (34) by $e^{k\Delta\phi/2}$ directly yields the Eqs. (23), note we must replace k by the usual complex number i, where $i^2 = -1$.

5 Design Equations for the Planar nR Chain

The goal of our design problem is to determine the dimensions of the planar nR chain that can position a tool held by its end-effector in a given set of task positions. The location of the base of the robot, the position of the tool frame, as well as the link dimensions and joint angles are considered to be design variables.

5.1 Relative Kinematics Equations for Specified Task Positions. Identify a set of planar task positions $[P_j]$, $j = 1, \ldots, m$. Then, the physical dimensions of the chain are defined by the requirement that for each position $[P_j]$ there is a joint parameter vector $\vec{\theta}_j$ such that the kinematics equations of the chain yield

$$[P_j] = [D(\vec{\theta}_j)], \quad i = 1, \ldots, m \tag{35}$$

Now, choose $[P_1]$ as the reference position and compute the relative displacements $[P_j][P_1^{-1}] = [P_{1j}]$, $j = 2, \ldots, m$. This formulation of the linkage design equations can be found in Suh and Racliffe [22]. The result is the relative kinematics equations

$$[P_{1j}] = e^{\Delta\theta_{1j}J_1}e^{\Delta\theta_{2j}J_2}\ldots e^{\Delta\theta_{nj}J_n}, \quad j = 2, \ldots, m \tag{36}$$

where

$$\Delta\vec{\theta}_j = \vec{\theta}_j - \vec{\theta}_1 = (\Delta\theta_{1j}, \ldots, \Delta\theta_{nj})$$

The complex number form of (36) yields the equations

$$e^{i\Delta\phi_j} = e^{i(\Delta\theta_{1j} + \Delta\theta_{2j} + \ldots + \Delta\theta_{nj})}$$

$$\begin{aligned}(1 - e^{i\Delta\phi_j})\mathbf{P}_{1j} &= (1 - e^{i\Delta\theta_{1j}})\mathbf{C}_1 + e^{i\Delta\theta_{1j}}(1 - e^{i\Delta\theta_{2j}})\mathbf{C}_2 + \ldots \\ &+ e^{i(\Delta\theta_{1j} + \Delta\theta_{2j} + \ldots + \Delta\theta_{n-1,j})}(1 - e^{i\Delta\theta_{nj}})\mathbf{C}_n, \quad j = 2, \ldots, m\end{aligned} \tag{37}$$

where $\Delta\phi_j = \phi_j - \phi_1$ and $\mathbf{P}_{1j}$ is the pole of the relative displacement $[P_{1j}]$. These are the equations we use to design the planar nR chain.

In terms of elements of the Clifford algebra we obtain the equivalent set of design equations

$$e^{k\Delta\phi_j/2} = e^{k(\Delta\theta_{1j} + \Delta\theta_{2j} + \cdots + \Delta\theta_{nj})/2}$$

$$\begin{aligned}\sin\tfrac{\Delta\phi_j}{2}\mathbf{P}_{1j} &= \sin\tfrac{\Delta\theta_{1j}}{2}\mathbf{C}_1 e^{-k(\Delta\theta_{2j} + \ldots + \Delta\theta_{nj})/2} \\ &+ e^{k\Delta\theta_{1j}/2}\sin\tfrac{\Delta\theta_{2j}}{2}\mathbf{C}_2 e^{-k(\Delta\theta_{3j} + \ldots + \Delta\theta_{nj})/2} + \ldots \\ &+ e^{k(\Delta\theta_{1j} + \Delta\theta_{2j} + \ldots + \Delta\theta_{n-1,j})/2}\sin\tfrac{\Delta\theta_{nj}}{2}\mathbf{C}_n, \quad j = 2, \ldots, m\end{aligned} \tag{38}$$

The Eqs. (38) allow the introduction of $\sin(\Delta\theta_{ij}/2)$ and $\cos(\Delta\theta_{ij}/2)$ as algebraic unknowns so these equations can be solved for the various joint angles as well as the coordinates of the joints. This is demonstrated below in our algebraic solution of the five position synthesis of a 2R chain.

5.2 The Number of Design Positions and Free Parameters. If we specify m task positions, then Eqs. (37) provide $m-1$ rotation and $2(m-1)$ translation equations. The unknowns consist of the $n(m-1)$ relative joint angles, and the $2n$ coordinates $\mathbf{C}_i$, $i = 1, \ldots, n$.

It is useful to notice that the rotation equations are solved independently, which means that they determine $m-1$ of the relative joint angles. Thus, we have $2(m-1)$ translation equations to solve for $(n-1)(m-1)$ joint variables and $2n$ coordinates $\mathbf{C}_i$, that is

$$E = 2n + (n-1)(m-1) - 2(m-1) = m(n-3) + n + 3 \tag{39}$$

where E excess of unknowns over equations.

Notice that except for $n=1$ and $n=2$ the excess of variables over equations is greater than zero. For $n=1$, we see that $m=2$ yields an exact formula for what is equivalent to the pole of a relative displacement. For $n=2$, we find that an exact solution is possible for $m=5$, which is Burmester's result that a 2R chain can be designed to reach five specified positions (Burmester [23], Hartenberg and Denavit [24]).

Now consider the case $n=3$, which has six unknown coordinates $\mathbf{C}_i$, $i=1, 2, 3$, and $2(m-1)$ joint variables that are determined by $2(m-1)$ equations. The excess is $E=6$ no matter how many positions are specified. In order to formulate this design problem, we specify the $m-1$ relative joint angles around $\mathbf{C}_1$. This is equivalent to adding $m-1$ design equations, which means that (39) takes the form $E = 6 - (m-1)$. The result is that given seven positions, $m=7$, we obtain a set of equations that determine the six coordinates $\mathbf{C}_i$, $i=1,2,3$.

6 The Standard Form Equations

The synthesis of planar 2R chains is the primary step in the design of four-bar linkages, which are constructed by joining the end links of two 2R chains to form the floating link, or coupler. Specializing the relative kinematics Eqs. (37) to this case, we obtain

$$e^{i\Delta\phi_j} = e^{i(\Delta\theta_{1j} + \Delta\theta_{2j})}$$

$$(1 - e^{i\Delta\phi_j})\mathbf{P}_{1j} = (1 - e^{i\Delta\theta_{1j}})\mathbf{C}_1 + e^{i\Delta\theta_{1j}}(1 - e^{i\Delta\theta_{2j}})\mathbf{C}_2, \quad j = 2, \ldots, m \tag{40}$$

We now show that this is the standard form equation used by Sandor and Erdman [3] for planar mechanism synthesis.

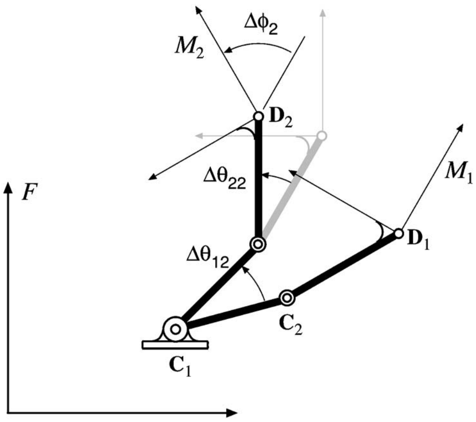

Fig. 4 Two positions of a planar 2R chain

The standard form equation is obtained by equating the relative displacement vector between two positions to the difference of vectors along the chain in the two positions. See Fig. 4. Let $\mathbf{C}_1$ be the fixed pivot and $\mathbf{C}_2$ the moving pivot when the tool frame of 2R chain is aligned with the first position.

Introduce the relative vectors $\mathbf{W}=\mathbf{C}_2-\mathbf{C}_1$ and $\mathbf{Z}=\mathbf{D}_1-\mathbf{C}_2$, where $\mathbf{D}_1$ is the translation vector to the first task position. We can now form the vector equations

$$\mathbf{D}_1 = \mathbf{C}_1 + \mathbf{W} + \mathbf{Z}$$

$$\mathbf{D}_2 = \mathbf{C}_1 + \mathbf{W}e^{i\Delta\theta_{12}} + \mathbf{Z}e^{i(\Delta\theta_{12}+\Delta\theta_{22})}$$

$$\cdots$$

$$\mathbf{D}_m = \mathbf{C}_1 + \mathbf{W}e^{i\Delta\theta_{1m}} + \mathbf{Z}e^{i(\Delta\theta_{1m}+\Delta\theta_{2m})} \tag{41}$$

Recall that multiplication by the complex exponential rotates a vector by an angle measured relative to the x axis of the fixed frame.

Subtract the first equation from the remaining m to obtain

$$\delta_{1j} = \mathbf{W}(e^{i\Delta\theta_{1j}} - 1) + \mathbf{Z}(e^{i(\Delta\theta_{1j}+\Delta\theta_{2j})} - 1), \quad j = 2, \ldots, m \tag{42}$$

where $\delta_{1j}=\mathbf{D}_j-\mathbf{D}_1$. Notice that the rotation of the jth task frame relative to the first position is

$$\Delta\phi_j = \Delta\theta_{1j} + \Delta\theta_{2j} \tag{43}$$

Sandor and Erdman [3] call Eqs. (42) the *standard form equation* and they use it to formulate a range of linkage synthesis problems based on the planar 2R chain.

Now substitute the definition of the relative vectors $\mathbf{W}$, $\mathbf{Z}$ and δ_{ij} back into the standard form equation to obtain

$$\mathbf{D}_j - \mathbf{D}_1 = (\mathbf{C}_2 - \mathbf{C}_1)(e^{i\Delta\theta_{1j}} - 1) + (\mathbf{D}_1 - \mathbf{C}_2)(e^{i(\Delta\theta_{1j}+\Delta\theta_{2j})} - 1)$$

and simplify to obtain

$$\mathbf{D}_j - \mathbf{D}_1 e^{i\Delta\phi_j} = (1 - e^{i\Delta\theta_{1j}})\mathbf{C}_1 + e^{i\Delta\theta_{1j}}(1 - e^{i\Delta\theta_{2j}})\mathbf{C}_2, \quad j = 1, \ldots, m \tag{44}$$

In order to show that this equation is identical to (40) we compute the pole $\mathbf{P}_{1j}$ in terms of the translation vectors $\mathbf{D}_j$ and $\mathbf{D}_1$

Let $[D_j]=[e^{i\phi_j},\mathbf{D}_j]$, $j=1,\ldots,m$, and compute

$$[D_{1j}] = [D_j][D_1]^{-1} = [e^{i(\phi_j-\phi_1)}, \mathbf{D}_j - \mathbf{D}_1 e^{i(\phi_j-\phi_1)}] \tag{45}$$

Now the pole $\mathbf{P}_{1j}$ of this relative displacement is defined as the point that has the same coordinates before and after the displacement, which means it satisfies the condition

$$\mathbf{P}_{1j} = e^{i(\phi_j-\phi_1)}\mathbf{P}_{1j} + \mathbf{D}_j - \mathbf{D}_1 e^{i(\phi_j-\phi_1)} \tag{46}$$

Thus, we obtain

$$(1 - e^{i\Delta\phi_j})\mathbf{P}_{1j} = \mathbf{D}_j - \mathbf{D}_1 e^{\Delta\phi_j} \tag{47}$$

and substituting this into (44), we find that the relative kinematics Eqs. (40) are exactly Sandor and Erdman's standard form equations.

7 Synthesis of 3R Serial Chains

The planar 3R robot has three degrees of freedom and can reach any set of positions within its workspace boundary. The design equations for m task positions take the form

$$e^{i\Delta\phi_j} = e^{i(\Delta\theta_{1j}+\Delta\theta_{2j}+\Delta\theta_{3j})}$$

$$(1 - e^{i\Delta\phi_j})\mathbf{P}_{1j} = (1 - e^{i\Delta\theta_{1j}})\mathbf{C}_1 + e^{i\Delta\theta_{1j}}(1 - e^{i\Delta\theta_{2j}})\mathbf{C}_2 + e^{i(\Delta\theta_{1j}+\Delta\theta_{2j})}(1 - e^{i\Delta\theta_{3j}})\mathbf{C}_3 \quad j = 2, \ldots, m, \tag{48}$$

We consider the design of this chain for three, five, and seven task positions with the condition that the relative joint angles around $\mathbf{C}_1$ are specified by the designer.

7.1 Three Task Positions. If we specify three task positions, the result is four translation design equations, or two complex equations, which determine the six coordinates of $\mathbf{C}_i$ and the $2(3-1)=4$ relative joint angles around $\mathbf{C}_1$ and $\mathbf{C}_2$. The joint angles around $\mathbf{C}_3$ are determined by the rotation design equations.

If we specify the four unknown relative joint angles and $\mathbf{C}_1$, then these four design equations are linear in the coordinates of $\mathbf{C}_2$ and $\mathbf{C}_3$. The result is two complex linear equations in two complex unknowns

$$\kappa_{12} = e^{i\Delta\theta_{12}}(1 - e^{i\Delta\theta_{22}})\mathbf{C}_2 + e^{i(\Delta\theta_{12}+\Delta\theta_{22})}(1 - e^{i\Delta\theta_{32}})\mathbf{C}_3$$

$$\kappa_{13} = e^{i\Delta\theta_{13}}(1 - e^{i\Delta\theta_{23}})\mathbf{C}_2 + e^{i(\Delta\theta_{13}+\Delta\theta_{23})}(1 - e^{i\Delta\theta_{33}})\mathbf{C}_3 \tag{49}$$

where κ_{1j} are the known complex numbers

$$\kappa_{1j} = (1 - e^{i\Delta\phi_j})\mathbf{P}_{1j} - (1 - e^{i\Delta\theta_{1j}})\mathbf{C}_1 \tag{50}$$

7.2 Five Task Positions. If five task positions are specified, then we have eight translation design equations in 14 unknowns, the six coordinates $\mathbf{C}_i$ and eight relative joint angles. Now specify the coordinates of $\mathbf{C}_1$ and the four relative angles around it to define six parameters. The result is the four complex equations

$$\kappa_{12} = e^{i\Delta\theta_{12}}(1 - e^{i\Delta\theta_{22}})\mathbf{C}_2 + e^{i(\Delta\theta_{12}+\Delta\theta_{22})}(1 - e^{i\Delta\theta_{32}})\mathbf{C}_3$$

$$\cdots$$

$$\kappa_{15} = e^{i\Delta\theta_{15}}(1 - e^{i\Delta\theta_{25}})\mathbf{C}_2 + e^{i(\Delta\theta_{15}+\Delta\theta_{25})}(1 - e^{i\Delta\theta_{35}})\mathbf{C}_3 \tag{51}$$

where κ_{1j} are known complex number defined by (50). These equations have exactly the same structure as Sandor and Erdman's standard form Eqs. (44) for five position synthesis and are solved in the same way.

7.3 Seven Task Positions. If seven task positions are specified as well as the six relative joint angles around $\mathbf{C}_1$, then we obtain 12 translation design equations in the 12 unknowns consisting of the six joint coordinates $\mathbf{C}_i$ and six relative joint angles around $\mathbf{C}_2$. The result is six complex equations

$$(1 - e^{i\Delta\phi_2})\mathbf{P}_{12} = (1 - e^{i\Delta\theta_{12}})\mathbf{C}_1 + e^{i\Delta\theta_{12}}(1 - e^{i\Delta\theta_{22}})\mathbf{C}_2 + e^{i(\Delta\theta_{12}+\Delta\theta_{22})} \times (1 - e^{i\Delta\theta_{32}})\mathbf{C}_3$$

$$\cdots$$

$$(1 - e^{i\Delta\phi_7})\mathbf{P}_{17} = (1 - e^{i\Delta\theta_{17}})\mathbf{C}_1 + e^{i\Delta\theta_{17}}(1 - e^{i\Delta\theta_{27}})\mathbf{C}_2 + e^{i(\Delta\theta_{17}+\Delta\theta_{27})} \times (1 - e^{i\Delta\theta_{37}})\mathbf{C}_3 \tag{52}$$

This problem has been solved using matrix resultants by Lin and Jia [25] and using homotopy continuation by Subbian and Flugrad [6].

8 Single DOF Coupled Serial Chains

Krovi et al. [7] expand the standard form equations to nR chains in which the joints are coupled by cable transmissions so the system has one degree of freedom. They call the chain a *single degree-of-freedom coupled serial chain*. We formulate an equivalent form of their design equations using the relative kinematics Eqs. (37).

Consider a planar nR serial chain in which each joint is connected to ground through a series of cables and pulleys located at each joint. Let each pulley have the same diameter and the cables routed through the links so they form parallelogram linkages. The result is n drive pulleys at the base of the chain that control the angle α_i of the ith link relative to the-x axis of the world frame, which means each joint angle is given by

$$\theta_i = \alpha_i - \alpha_{i-1} \tag{53}$$

We now introduce a single drive angle β such that each joint angle is given by relation $\theta_i = R_i\beta$, where R_i denotes a constant speed ratio. The relations (53) yield the transmission matrix $[C]$ to the base drive angles are given by

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{Bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & & \cdots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} R_1 \\ R_2, \\ \vdots \\ R_n \end{bmatrix} \beta \tag{54}$$

or

$$\vec{\alpha} = [C][R]\beta \tag{55}$$

where $[R]$ is the column matrix formed by the speed ratios. Our formulation differs slightly from Krovi et al. [7] in that we have added the drive variable β and therefore an additional speed ratio R_1.

Consider the design of an nR chain in which the speed ratios R_i, $i=1\ldots,n$, are specified. Substitute these speed ratios into the rotation term of the design Eqs. (37) to obtain

$$e^{i\Delta\phi_j} = e^{i(R_1+R_2+\ldots+R_m)\Delta\beta_j}, \quad j=2,\ldots,m \tag{56}$$

where $\Delta\beta_j = \beta_j - \beta_1$ is the relative rotation of the drive angle. We find for each relative task position that

$$\Delta\beta_j = \frac{\Delta\phi_j}{R_1 + R_2 + \ldots + R_n} \tag{57}$$

Substitute this into the translation terms of (37) to define a linear equation in the coordinates $\mathbf{C}_i$, $i=1,\ldots,n$ for each relative task position

$$\begin{aligned}(1-e^{i\Delta\phi_j})\mathbf{P}_{1j} = {} & (1-e^{iR_1\Delta\beta j})\mathbf{C}_1 + e^{iR_1\Delta\beta j}(1-e^{iR_2\Delta\beta j})\mathbf{C}_2 + \ldots \\ & + e^{i(R_1+R_2+\ldots+R_{n-1})\Delta\beta j}(1-e^{iR_n\Delta\beta_j})\mathbf{C}_n, \quad j=2,\ldots,m\end{aligned} \tag{58}$$

Given $m=n+1$ task positions, we can solve these equations for the n complex unknowns $\mathbf{C}_i$. The result is a coupled serial nR chain designed to reach $n+1$ arbitrarily specified task positions.

9 Algebraic Solution of the RR Design Equations

In this section, we solve the standard form equations for five position synthesis using the Clifford algebra design Eqs. (38), given by

$$\begin{aligned}-\sin\tfrac{\Delta\phi_j}{2}\mathbf{P}_{1j}j\epsilon + e^{k\Delta\phi_j/2} &= \left(-\sin\tfrac{\Delta\theta_{1j}}{2}\mathbf{C}_1 j\epsilon + e^{k\Delta\theta_{1j}/2}\right) \\ &\quad\times\left(-\sin\tfrac{\Delta\theta_{2j}}{2}\mathbf{C}_2 j\epsilon + e^{k\Delta\theta_{2j}/2}\right) \quad j=2,3,4,5\end{aligned} \tag{59}$$

For convenience we relabel the coordinates of the fixed and moving pivots so that $\mathbf{C}_1=\mathbf{G}=(g_x,g_y)$ and $\mathbf{C}_2=\mathbf{W}=(w_x,w_y)$. Similarly, label the joint angles $\theta_1=\beta$ and $\theta_2=\alpha$, as shown in Fig. 1. Then we have

$$\begin{aligned}-\sin\tfrac{\Delta\phi_j}{2}\mathbf{P}_{1j}j\epsilon + e^{k\Delta\phi_j/2} &= \left(-\sin\tfrac{\Delta\beta_j}{2}\mathbf{G}j\epsilon + e^{k\Delta\beta_j/2}\right) \\ &\quad\times\left(-\sin\tfrac{\Delta\alpha_j}{2}\mathbf{W}j\epsilon + e^{k\Delta\alpha_j/2}\right) \quad j=2,3,4,5\end{aligned} \tag{60}$$

9.1 Matrix Form of the Design Equations. Expand and collect the components of the Clifford algebra elements in the design Eqs. (60) to define the arrays

$$\begin{Bmatrix} s\frac{\Delta\phi_j}{2}p_{xj} \\ s\frac{\Delta\phi_j}{2}p_{yj} \\ s\frac{\Delta\phi_j}{2} \\ c\frac{\Delta\phi_j}{2} \end{Bmatrix} = \begin{Bmatrix} s\frac{\Delta\beta_j}{2}c\frac{\Delta\alpha_j}{2}g_y + c\frac{\Delta\beta_j}{2}s\frac{\Delta\alpha_j}{2}w_y - s\frac{\Delta\beta_j}{2}s\frac{\Delta\alpha_j}{2}(g_x - w_x) \\ -s\frac{\Delta\beta_j}{2}c\frac{\Delta\alpha_j}{2}g_x - c\frac{\Delta\beta_j}{2}s\frac{\Delta\alpha_j}{2}w_x - s\frac{\Delta\beta_j}{2}s\frac{\Delta\alpha_j}{2}(g_y - w_y) \\ s\frac{\Delta\beta_j}{2}c\frac{\Delta\alpha_j}{2} + c\frac{\Delta\beta_j}{2}s\frac{\Delta\alpha_j}{2} \\ c\frac{\Delta\beta_j}{2}c\frac{\Delta\alpha_j}{2} - s\frac{\Delta\beta_j}{2}s\frac{\Delta\alpha_j}{2} \end{Bmatrix}$$

$$j=2,3,4,5 \tag{61}$$

where s and c denote the sine and cosine functions. This equation can be written in the matrix form

$$\begin{Bmatrix} s\frac{\Delta\phi_j}{2}p_{xj} \\ s\frac{\Delta\phi_j}{2}p_{yj} \\ s\frac{\Delta\phi_j}{2} \\ c\frac{\Delta\phi_j}{2} \end{Bmatrix} = \begin{bmatrix} g_y & w_y & w_x - g_x & 0 \\ -g_x & -w_x & w_y - g_y & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \times \begin{Bmatrix} s\frac{\Delta\beta_j}{2}c\frac{\Delta\alpha_j}{2} \\ c\frac{\Delta\beta_j}{2}s\frac{\Delta\alpha_j}{2} \\ s\frac{\Delta\beta_j}{2}s\frac{\Delta\alpha_j}{2} \\ c\frac{\Delta\beta_j}{2}c\frac{\Delta\alpha_j}{2} \end{Bmatrix} \quad j=2,3,4,5 \tag{62}$$

This matrix equation can be inverted to define the joint variables in terms of the joint coordinates $\mathbf{G}$ and $\mathbf{W}$, that is

$$\frac{1}{R^2}\begin{bmatrix} -(w_y-g_y) & w_x-g_x & \mathbf{W}\cdot(\mathbf{W}-\mathbf{G}) & 0 \\ w_y-g_y & -(w_x-g_x) & -\mathbf{G}\cdot(\mathbf{W}-\mathbf{G}) & 0 \\ w_x-g_x & w_y-g_y & g_xw_y-g_yw_x & 0 \\ w_x-g_x & w_y-g_y & g_xw_y-g_yw_x & R^2 \end{bmatrix}\begin{Bmatrix} s\frac{\Delta\phi_j}{2}p_{xj} \\ s\frac{\Delta\phi_j}{2}p_{yj} \\ s\frac{\Delta\phi_j}{2} \\ c\frac{\Delta\phi_j}{2} \end{Bmatrix} = \begin{Bmatrix} s\frac{\Delta\beta_j}{2}c\frac{\Delta\alpha_j}{2} \\ c\frac{\Delta\beta_j}{2}s\frac{\Delta\alpha_j}{2} \\ s\frac{\Delta\beta_j}{2}s\frac{\Delta\alpha_j}{2} \\ c\frac{\Delta\beta_j}{2}c\frac{\Delta\alpha_j}{2} \end{Bmatrix} \tag{63}$$

where R is the distance between the two joints, that is $R^2 = (\mathbf{W} - \mathbf{G})^2$. This solves the inverse kinematics problem for the 2R

Table 1 Five precision positions

Position	Point	Angle (deg)
1	(0,0)	0
2	(−0.5,4)	5
3	(−1.5,5)	5
4	(−2.0,5.5)	60
5	(−2.5,5)	120

chain.

The joint variables can be eliminated from this equation by using the identity

$$\frac{\sin\frac{\Delta\beta_j}{2}\sin\frac{\Delta\alpha_j}{2}}{\cos\frac{\Delta\beta_j}{2}\sin\frac{\Delta\alpha_j}{2}}=\frac{\sin\frac{\Delta\beta_j}{2}\cos\frac{\Delta\alpha_j}{2}}{\cos\frac{\Delta\beta_j}{2}\cos\frac{\Delta\alpha_j}{2}} \tag{64}$$

The result is a set of four quadratic design equations

$$\begin{aligned}\mathcal{R}_j\colon\ &(g_xw_y-g_yw_x)c\frac{\Delta\phi_j}{2}+(g_xw_x+g_yw_y)s\frac{\Delta\phi_j}{2}+g_x\Big(-p_{xj}c\frac{\Delta\phi_j}{2}\\&+p_{yj}s\frac{\Delta\phi_j}{2}\Big)+g_y\Big(-p_{yj}c\frac{\Delta\phi_j}{2}-p_{xj}s\frac{\Delta\phi_j}{2}\Big)+w_x\Big(p_{xj}c\frac{\Delta\phi_j}{2}+p_{yj}s\frac{\Delta\phi_j}{2}\Big)\\&+w_y\Big(p_{yj}c\frac{\Delta\phi_j}{2}-p_{xj}s\frac{\Delta\phi_j}{2}\Big)+(p_{xj}^2+p_{yj}^2)s\frac{\Delta\phi_j}{2}\\&=0\quad j=2,3,4,5\end{aligned} \tag{65}$$

These equations are linear in the unknowns w_x and w_y, so we can collect their coefficients to define the linear equations

$$\begin{bmatrix}A_2 & B_2 & C_2\\ \vdots & \vdots & \vdots\\ A_5 & B_5 & C_5\end{bmatrix}\begin{Bmatrix}w_x\\ w_y\\ 1\end{Bmatrix}=\begin{Bmatrix}0\\ \vdots\\ 0\end{Bmatrix} \tag{66}$$

where

$$A_j=(p_{xj}-g_y)c\frac{\Delta\phi_j}{2}+(p_{yj}+g_x)s\frac{\Delta\phi_j}{2}$$

$$B_j=(p_{yj}+g_x)c\frac{\Delta\phi_j}{2}+(-p_{xj}+g_y)s\frac{\Delta\phi_j}{2}$$

$$C_j=(p_{xj}^2+p_{yj}^2)s\frac{\Delta\phi_j}{2}-(g_xp_{xj}+g_yp_{yj})c\frac{\Delta\phi_j}{2}+(g_xp_{yj}-g_yp_{xj})s\frac{\Delta\phi_j}{2} \tag{67}$$

Let M_5 be the 3×3 matrix obtained from (66) by removing row $j=5$. Similarly, let M_4, M_3, and M_2 be the 3×3 matrices be obtained in the same way by removing the specified rows. If the determinants of these four matrices are zero, then the linear Eqs. (66) have rank 2, and they can be solved to determine w_x and w_y.

Denote (g_x,g_y) as (x,y) so that each of the determinants defines a cubic curve $p_j(x,y)=\det(M_j)=0$, often called a "centerpoint curve." We can collect the coefficients of x in these four polynomials, so we have

$$\begin{aligned}p_j(x,y)&=a_jgx^3+(b_jy+c_j)x^2+(a_jy^2+d_jy+e_j)x+(b_jy^3+f_jy^2\\&\quad+h_jy+k_j)=0\quad j=2,3,4,5\end{aligned} \tag{68}$$

McCarthy [27] expands these coefficients and shows that this curve has the structure of a "circular cubic," which is a planar cubic curve that passes through the circle points at infinity.

Assemble these equations into the linear system

Table 2 The two real solutions yields two candidate designs

Design	Fixed pivot	Moving pivot
1	(−2.188,5.085)	(2.655, 3.864)
2	(−4.738,1.312)	(−0.852,−0.207)

$$\begin{bmatrix}a_2 & (b_2y+c_2) & (a_2y^2+d_2y+e_2) & (b_2y^3+f_2y^2+h_2y+k_2)\\ \vdots & \vdots & \vdots & \vdots\\ a_5 & (b_5y+d_5) & (a_5y^2+d_5y+e_5) & (b_5y^3+f_5y^2+h_5y+k_5)\end{bmatrix}\begin{Bmatrix}x^3\\ x^2\\ x\\ 1\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\end{Bmatrix} \tag{69}$$

or

$$[N]\mathbf{v}=0 \tag{70}$$

where $\mathbf{v}=(x^3,x^2,x,1)$. These cubic curves have a simultaneous solution if the determinant of the coefficient matrix $[N]$ is zero, that is

$$\det(N)=n_4y^4+n_3y^3+n_2y^2+n_1y+n_0=0 \tag{71}$$

This is a univariate polynomial of degree four that yields as many as four solutions for $y=g_y$.

Substitute these values into (69) to find $x=g_x$, thus defining the pivot $\mathbf{G}=(g_x,g_y)$. For each of these pivots, we can compute an associated point $\mathbf{W}=(w_x,w_y)$ by solving (66). The result is as many as four combinations of $\mathbf{G}$ and $\mathbf{W}$ that define 2R chains that reach the five specified task positions.

9.2 Example of Planar RR Five Position Synthesis. We illustrate the elimination methodology for the RR chain with an example from Sandor and Erdman [3] for the design of a four-bar mechanism to guide a bucket through the set of positions shown in Table 1.

Choose the first position as the reference configuration and compute the relative displacements, expressed as elements of the Clifford algebra

$$\begin{aligned}P_{12}&=-0.163i\epsilon+2.01j\epsilon+0.0436k+0.999\\P_{13}&=-0.640i\epsilon+2.53j\epsilon+0.0436k+0.999\\P_{14}&=0.509i\epsilon+2.88j\epsilon+0.500k+0.866\\P_{15}&=1.54i\epsilon+2.33j\epsilon+0.866k+0.500.\end{aligned} \tag{72}$$

We substitute these values in Eq. (65) to obtain four design equations in the coordinates of the pivots. Following the elimination procedure described above, we obtain the univariate polynomial

$$\det(N)\colon\ y^4-10.33y^3+550.72y^2-3345.39y+3461.02=0 \tag{73}$$

The polynomial yields four roots out of which two are complex.

Table 3 Inverse kinematics results for the two candidate designs

Design	Angle β (rad)	Angle α (rad)
1	(−0.25,0.64,0.93,1.9,3.1)	(−1.93,−2.7,−3.0,−3.1,−3.2)
2	(−0.37,0.62,0.96,0.93,0.82)	(0.61,−0.29,−0.63,0.35,1.5)

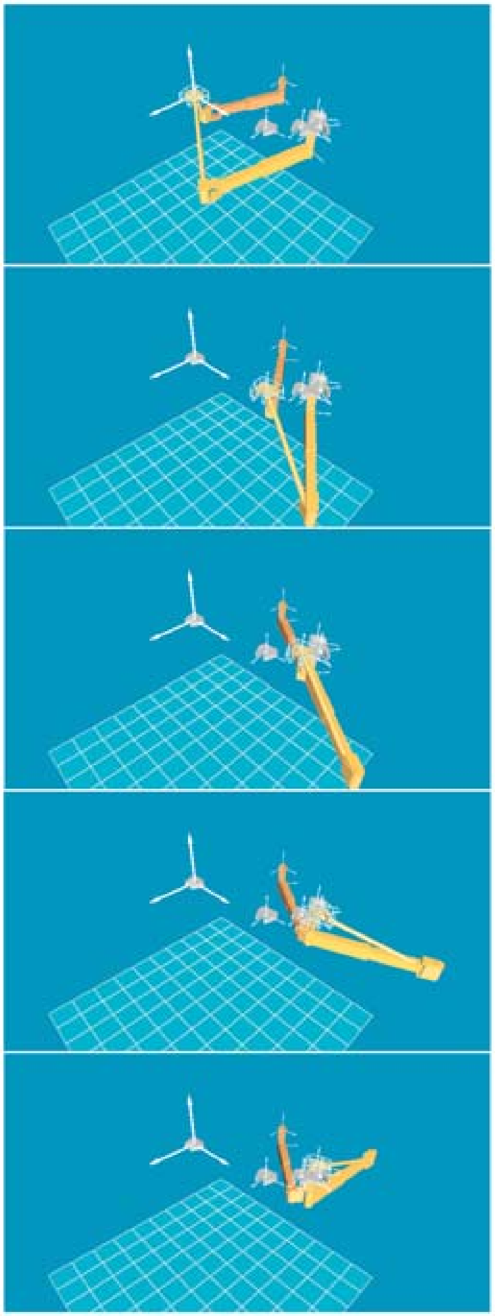

Fig. 5 The two 2R design candidates reaching each of the five specified task positions

We use the real roots to solve linearly for the rest of the coordinates of the pivots. We obtain the two solutions shown in Table 2.

The values of the pivots are substituted in Eq. (62) to compute the inverse kinematics of the robot. The system yields the relative angles; adding the initial angles to reach the reference position, we obtain the set of joint angles presented in Table 3 for both solutions. Figure 5 shows both solutions reaching the five positions.

Table 4 Seven precision positions

Position	Point	Angle (deg)	Angle $\theta_{1i}-\theta_{10}$ (deg)
1	(1, 0)	0	0
2	(1.01, 0.48)	25.71	50
3	(0.78, 0.98)	51.43	100
4	(0.31, 1.36)	77.14	150
5	(−0.35, 1.53)	102.86	200
6	(−1.09, 1.37)	128.57	250
7	(−1.77, 0.85)	154.29	300

Table 5 A solution RRR chain

	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_3$
Pivots	(−0.03, 0.00)	(0.07, 0.56)	(−0.12, −0.66)
Angles (deg)	80.18	−179.01	129.27

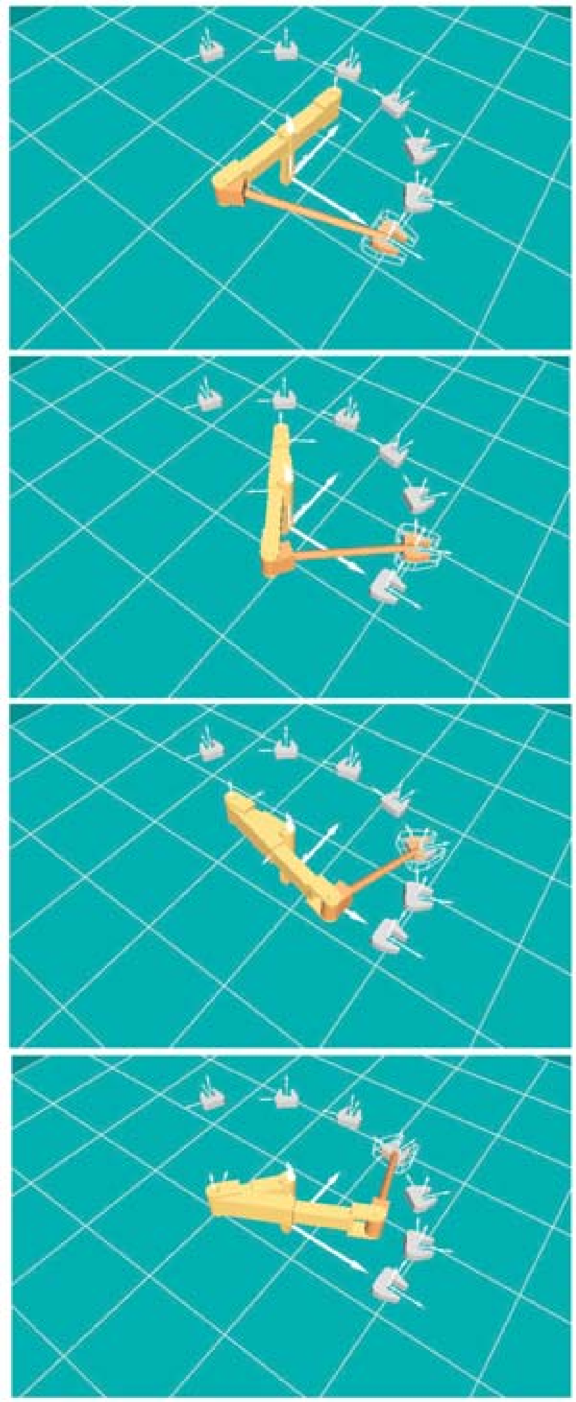

Fig. 6 RRR design reaching positions 1, 2, 3, and 4

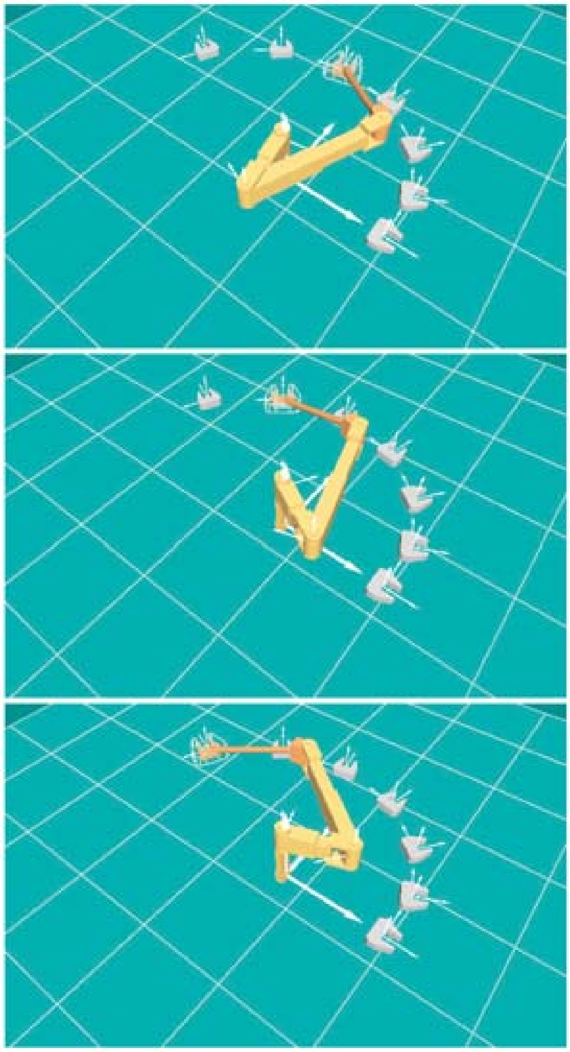

Fig. 7 RRR design reaching positions 5, 6, and 7

10 Synthesis of a Planar 3R Chain

Here we show how to formulate the design equations of an RRR planar chain. Consider the task of following the logarithmic spiral $r=a\exp\phi$, with $a=0.25$. In order to solve the design equations, the input crank angle θ_1 must also be specified in each of the task positions. The seven positions, as well as the prescribed values for θ_1, are given in Table 4.

Equating the Clifford algebra relative kinematics equations to the set of relative positions, we obtain a set of $4\cdot 6=24$ equations

$$e^{i(\Delta\phi_j-\Delta\theta_{1j})}=e^{i(\Delta\theta_{2j}+\Delta\theta_{3j})}$$

$$(1-e^{i\Delta\phi_j})\mathbf{P}_{1j}=(1-e^{i\Delta\theta_{1j}})\mathbf{C}_1+e^{i\Delta\theta_{1j}}(1-e^{i\Delta\theta_{2j}})\mathbf{C}_2+e^{i(\Delta\theta_{1j}+\Delta\theta_{2j})}(1-e^{i\Delta\theta_{3j}})\mathbf{C}_3 \quad j=2,\ldots,7 \tag{74}$$

The orientation equations for each position determine the joint angles $\Delta\theta_{2j}$ as a function of the $\Delta\theta_{3j}$, which are then eliminated from the translation equations. The result is a set of twelve design equations in twelve unknowns, two for each task position, given by

$$-P_{1j}^x+A_1c_{1x}+A_2c_{1y}+A_3c_{2x}+A_4c_{2y}+A_5c_{3x}+A_6c_{3y}+(A_5c_{2x}+A_6c_{2y}-A_5c_{3x}-A_6c_{3y})\cos\Delta\theta_{2j}+(-A_6c_{2x}+A_5c_{2y}+A_6c_{3x}-A_5c_{3y})\sin\Delta\theta_{2j}=0$$

$$-P_{1j}^y-A_2c_{1x}+A_1c_{1y}-A_4c_{2x}+A_3c_{2y}-A_6c_{3x}+A_5c_{3y}+(A_6c_{2x}+A_5c_{2y}+A_6c_{3x}-A_5c_{3y})\cos\Delta\theta_{2j}+(-A_5c_{2x}-A_6c_{2y}+A_5c_{3x}+A_6c_{3y})\sin\Delta\theta_{2j}=0, \quad j=2,\ldots,7 \tag{75}$$

The components P_{ij}^x and P_{ij}^y of the Clifford algebra element for position P_{ij} are known, so we determine the coordinates defining the three joints, $\mathbf{C}_1$, $\mathbf{C}_2$, and $\mathbf{C}_3$, in the reference configuration, as well as the values of the angle θ_2 to reach each task position. This is done numerically using using Mathematica. Table 5 presents one of the solutions obtained in the reference configuration.

Figures 6 and 7 show the RRR design reaching the seven positions with the prescribed values for θ_1.

11 Summary

This paper derives design equations for planar serial chains directly from the relative kinematics equations of the chain. The well-known standard form equations used for the synthesis of planar 2R, 3R, and coupled serial chains are shown to be obtained in this way.

The exponential in the Clifford algebra $C^{+}(P^2)$ is introduced to provide an efficient formulation for the relative kinematics equations. The hypercomplex numbers in $C^{+}(P^2)$ are both a generalization of standard complex numbers and a specialization of dual quaternions. They yield a compact set of design equations and a convenient algebraic form for the joint angles. Examples demonstrate the use of these equations for the synthesis of 2R and 3R planar serial chains.

The result is a systematic derivation of the design equations for planar serial chains that can be generalized to spatial serial chains. In addition, the Clifford algebra formulation links this synthesis theory to the kinematic mapping techniques that have been used to study planar and spatial displacements.

Acknowledgments

The authors gratefully acknowledge the support of the National Science Foundation Grant No. DMII-0218285, as well as the helpful advice of the reviewers.

References

[1] Craig, J. J., 1989, *Introduction to Robotics, Mechanics and Control*, Addison Wesley, Reading, MA.

[2] Tsai, L.-W., 1999, *Robot Analysis: The Mechanics of Serial and Parallel Manipulators*, Interscience, New York.

[3] Sandor, G. N., and Erdman, A. G., 1984, *Advanced Mechanism Design: Analysis and Synthesis*, Prentice-Hall, Englewood Cliffs, NJ, Vol. 2.

[4] Lin, C.-S., and Erdman, A. G., 1987, "Dimensional Synthesis of Planar Triads for Six Positions," Mech. Mach. Theory, **22**, pp. 411–419.

[5] Chase, T. R., Erdman, A. G., and Riley, D. R., 1987, "Triad Synthesis for up to 5 Design Positions with Applications to the Design of Arbitrary Planar Mechanisms," ASME J. Mech., Transm., Autom. Des., **109**(4), pp. 426–434.

[6] Subbian, T., and Flugrad, D. R., 1994, "6 and 7 Position Triad Synthesis using Continuation Methods," ASME J. Mech. Des., **116**(2), pp. 660–665.

[7] Krovi, V., Ananthasuresh, G. K., and Kumar, V., 2002, "Kinematic and Kinetostatic Synthesis of Planar Coupled Serial Chain Mechanisms," ASME J. Mech. Des., **124**(2), pp. 301–312.

[8] Balli, S. S., and Chand, S., 2004, "Synthesis of a Five-Bar Mechanism of Variable Topology Type with Transmission Angle Control," ASME J. Mech. Des., **126**(1), pp. 128–134.

[9] Holte, J. E., Chase, T. R., and Erdman, A. G., 2000, "Mixed Exact-Approximate Position Synthesis of Planar Mechanisms," ASME J. Mech. Des., **122**(3), pp. 278–286.

[10] Mavroidis, C., Lee, E., and Alam, M., 2001, "A New Polynomial Solution to the Geometric Design Problem of Spatial RR Robot Manipulators Using the Denavit-Hartenberg Parameters," ASME J. Mech. Des., **123**(1), pp. 58–67.

[11] Lee, E., and Mavroidis, D., 2002, "Solving the Geometric Design Problem of Spatial 3R Robot Manipulators Using Polynomial Homotopy Continuation," ASME J. Mech. Des., **124**(4), pp. 652–661.

[12] Perez, A., 2003, "Dual Quaternion Synthesis of Constrained Robotic Systems," Ph.D. dissertation, Department of Mechanical and Aerospace Engineering, University of California, Irvine.

[13] Bottema, O., and Roth, B., 1979, *Theoretical Kinematics*, North Holland Press, New York.

[14] DeSa, S., and Roth, B., 1981, "Kinematic Mappings. Part2: Rational Algebraic Motions in the Plane," ASME J. Mech. Des., **103**, pp. 712–717.

[15] Ravani, B., and Roth, B., 1983, "Motion Synthesis Using Kinematic Mapping," ASME J. Mech., Transm., Autom. Des., **105**(3), pp. 460–467.
[16] Larochelle, P., and McCarthy, J. M., 1995, "Designing Planar Mechanisms using a Bi-invariant Quaternion Metric," ASME J. Mech. Des., **117**(4), pp. 646–651.
[17] Murray, A., Pierrot, F., Dauchez, P., and McCarthy, J., 1996, "On the Design of Parallel Manipulators for a Prescribed Workspace: a Planar Quaternion Approach," *5th International Symposium on Advances in Robot Kinematics*, June.
[18] McCarthy, J. M., 1990, *Introduction to Theoretical Kinematics*, MIT Press, Cambridge, MA.
[19] Murray, R. M., Sastry, S. S., and Li, Z., 1994, *A Mathematical Introduction to Robotic Manipulation*, CRC Press, Inc., Boca Raton, FL, p. 480.
[20] Gupta, K. C., 1986, "Kinematic analysis of manipulators using the zero reference position description," Int. J. Robot. Res., **5**(2), pp. 5–13.
[21] McCarthy, J. M., 1993, "Dual Quaternions and the Pole Triangle," *Modern Kinematics. Developments in the Last* 40 *Years*, Arthur G. Erdman, Ed., Wiley, New York.
[22] Suh, C. H., and Radcliffe, C. W., 1978, *Kinematics and Mechanisms Design*, Wiley, New York.
[23] Burmester, L., 1886, *Lehrbuch der Kinematik*, Verlag Von Arthur Felix, Leipzig.
[24] Hartenberg, R., and Denavit, J., 1964, *Kinematic Synthesis of Linkages*, McGraw-Hill, New York.
[25] Lin, C.-S., and Jia, B.-P., 1992, "Use of Resultant in the Dimensional Synthesis of Linkage Components: Motion Generation with Prescribed Timing," *Proceedings of the 1992 ASME Design Technical Conferences*, DE-Vol. **46**, pp. 489–496.
[26] Perez, A., and McCarthy, J. M., 2004, "Dual Quaternion Synthesis of Constrained Robotic Systems," ASME J. Mech. Des., **126**(3), pp. 425–435.
[27] McCarthy, J. M., 2000, *Geometric Design of Linkages*, Springer-Verlag, New York.