Kinematic synthesis of spatial serial chains using Clifford algebra exponentials

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Abstract: This article presents a formulation of the design equations for a spatial serial chain that uses the Clifford algebra exponential form of its kinematics equations. This is the even Clifford algebra $C^+(P^3)$, known as dual quaternions. These equations define the position and orientation of the end effector in terms of rotations or translations about or along the joint axes of the chain. Because the coordinates of these axes appear explicitly, specifying a set of task positions these equations can be solved to determine the location of the joints. At the same time, joint parameters or certain dimensions are specified to ensure that the resulting robotic system has specific features.

Keywords: kinematic synthesis, Clifford algebra exponentials, spatial serial robots

1 INTRODUCTION

This article presents the formulation of design equations for a spatial serial chain robot using the Clifford exponential form of its kinematics equations. This approach can be viewed as a generalization of the inverse kinematics problem, in which the locations of the joint axes, not just the joint angles, are computed to ensure that the chain can reach a specified set of task positions.

In what follows, the authors review the robot synthesis theory and then formulate the exponential form of the kinematics equations of a serial chain, which can be modified to form the Clifford algebra design equations. It is possible to count the number of structural and joint parameters to determine the number of positions needed to completely define the serial chain. The Clifford algebra form of these design equations provides a convenient algebraic structure, which is exploited in a numerical solver.

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2 LITERATURE REVIEW

The geometric design of a robot manipulator defines the topology and dimensions of the articulated system that provides the end-effector position and velocity performance needed for a specified set of applications. Herve [**1**] shows how to use the subgroups of the Lie group of rigid body displacements to formulate robotic systems with desired workspace properties. Wenger [**2**] describes the benefits of new serial chain topologies that allow reconfiguration within the workspace. Chablat *et al.* [**3**] and Li *et al.* [**4**] demonstrate different approaches to the design of specialized parallel platforms that optimize position and velocity performance, which seek a manipulator that has particular performance characteristics throughout its workspace.

A related approach to design seeks the robot manipulator that has specified position and velocity performance at precise locations within its workspace. Lee and Mavroidis [**5**, **6**] formulate and solve the design equations for a 3R spatial chain that reaches four arbitrarily specified positions – R denotes a revolute or hinged joint. This involves equating the kinematics equations of the chain to four selected positions and solving for the Denavit–Hartenberg parameters that satisfy these matrix equations. Perez and McCarthy [**7, 8**] formulate and solve the design equations for the RPC and RRP and related chains. This work builds on a tradition of spatial mechanism synthesis dating back to Suh [**9**] and Tsai and Roth [**10**] (see also references [**11–13**]).

The complexity of the geometric design problem increases with the number of structural parameters. Four independent parameters define the axis of a revolute joint and two define a prismatic joint; therefore, the spatial 5R chain has 20 structural parameters. Table 1 lists the five-degree-of-freedom (dof) chains and their associated permutations. It also lists the chains that are formed by combining the revolute and prismatic joints into spherical (S), cylindrical (C), and universal (T) joints – an RRR chain with concurrent axes forms an S joint, an RP chain with parallel axes forms a C joint, and an RR chain with perpendicular intersecting axes forms a T joint. A total of 126 topologies for five-dof serial chains are obtained. The design equations for the PPS, TS, CS, RPS, and RRS chains were originally formulated by Chen and Roth [**14**] and were recently solved by Su *et al.* [**15**] using polynomial homotopy. The most challenging is the RRS chain, which has 12 structural parameters and design equations of total degree 4 194 304 that yielded 42 615 solutions.

The goal of this article is a systematic formulation of the design equations for all serial chains. A benefit of the approach is that it can also be applied to the design of serial chains with six or more degrees of freedom.

3 KINEMATIC EQUATIONS OF A SERIAL CHAIN

The position and orientation of the end effector of a serial chain are defined in terms of its joint parameters and physical dimensions by the kinematics equations. The Denavit–Hartenberg formulation is used to assign the local joint coordinate frames (Fig. 1) necessary to define these equations [**16, 17**].

Let S_i, $i = 1, \ldots, n$, denote the n joint axes in the chain, which may define the axis of either a revolute or a prismatic joint. Introduce the line $A_{i,i+1}$ which is the common normal to the axes S_i and S_{i+1}. The

Table 1 The five-dof serial chains and the number of permutations

Chain	Permutations	Special cases	Permutations
PRPRP	10	CCP, CRPP, TPPP	19
RPRPR	10	CCR, PPS, TCP, TRPP, CRRP	36
RRRRP	5	CS, RPS, TCR, TRRP, CRRR, TTP	33
RRRRR	1	TS, RRS, TRRR, TTR	12
Total	26		100

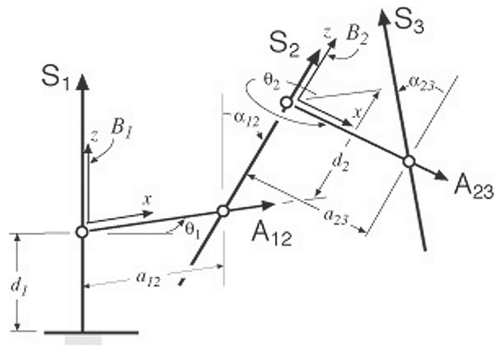

Fig. 1 Local frames for a serial chain

origin of the joint frame T_i is set at the intersection of S_i and $A_{i,i+1}$, such that the z-axis is S_i and the x-axis is $A_{i,i+1}$ [**18**].

This allows to write the kinematics equations of the chain in the form

$$\mathbf{D} = \mathbf{G}\mathbf{Z}(\theta_1, d_1)\mathbf{X}(\alpha_{12}, a_{12})\mathbf{Z}(\theta_2, d_2)\ldots \mathbf{X}(\alpha_{n-1,n}, a_{n-1,n})\mathbf{Z}(\theta_n, d_n)\mathbf{H} \quad (1)$$

where $\mathbf{Z}(\theta_i, d_i)$ and $\mathbf{X}(\alpha_{i,i+1}, a_{i,i+1})$ are the 4×4 homogeneous transforms

$$\mathbf{Z}(\theta_i, d_i) = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and}$$

$$\mathbf{X}(\alpha_{i,i+1}, a_{i,i+1})] = \begin{bmatrix} 1 & 0 & 0 & a_{i,i+1} \\ 0 & \cos\alpha_{i,i+1} & -\sin\alpha_{i,i+1} & 0 \\ 0 & \sin\alpha_{i,i+1} & \cos\alpha_{i,i+1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The parameters θ_i and d_i define the rotation of a revolute joint and the slide of a prismatic joint and $\alpha_{i,i+1}$ and $a_{i,i+1}$ define the dimensions of each link. Collectively, these are known as the Denavit–Hartenberg parameters. The transformation **G** locates the base of the robot in the world frame and **H** locates the tool frame relative to the last joint frame.

3.1 Product of exponentials

Rather than to use the Denavit–Hartenberg parameters for design, the 4×4 homogeneous transforms are written as matrix exponentials [**19**] so that the coordinates of the joint axes appear explicitly in the kinematics equations. The joint axes are expressed as lines using the Plücker coordinates. The Plücker coordinates of an axis S are given by

$$\mathsf{S} = (\boldsymbol{S}, \boldsymbol{S}^0) = (\boldsymbol{S}, \boldsymbol{C} \times \boldsymbol{S}) \quad (3)$$

where the first three-dimensional vector, $\boldsymbol{S}$, is a unit vector defining the direction of the axis and the second one, $\boldsymbol{S}^0$, is called the moment and is obtained as the cross product of a point on the axis, $\boldsymbol{C}$, and the direction $\boldsymbol{S}$. This is a particular case of a dual vector $\mathrm{V} = (\boldsymbol{V}, \boldsymbol{W})$, in which the direction is a unit vector and the pitch is zero. This is captured by the Plücker conditions below,

$$|\boldsymbol{S}| = 1 \quad \text{and} \quad \boldsymbol{S} \cdot (\boldsymbol{C} \times \boldsymbol{S}) = 0 \tag{4}$$

so that only four of the six coordinates defining an axis are independent.

Consider a displacement in which the moving body rotates the angle ϕ and slides the distance k around and along the screw axis $\mathrm{S} = (\boldsymbol{S}, \boldsymbol{C} \times \boldsymbol{S})$. Let $\mu = k/\phi$, then the screw $\mathrm{J} = (\boldsymbol{S}, \boldsymbol{V}) = (\boldsymbol{S}, \boldsymbol{C} \times \boldsymbol{S} + \mu \boldsymbol{S})$, where μ is called the pitch of the screw. The components of J define the 4×4 twist matrix

$$\mathbf{J} = \begin{bmatrix} 0 & -s_z & s_y & v_x \\ s_z & 0 & -s_x & v_y \\ -s_y & s_x & 0 & v_z \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5}$$

and the 4×4 homogeneous transform representing a rotation ϕ and a translation k about and along an axis S, $\mathbf{T}(\phi, k, \mathrm{S})$, is defined as the matrix exponential

$$\mathbf{T}(\phi, k, \mathrm{S}) = \mathrm{e}^{\phi \mathbf{J}} \tag{6}$$

The matrix exponential takes a simple form for the matrices $\mathbf{Z}(\theta_i, d_i)$ and $\mathbf{X}(\alpha_{i,i+1}, a_{i,i+1})$. The screws defined for these two transformations are $\mathrm{K} = (\boldsymbol{k}, \nu \boldsymbol{k})$ and $\mathrm{I} = (\boldsymbol{i}, \lambda \boldsymbol{i})$, where $\nu = d_i/\theta_i$ and $\lambda = a_{i,i+1}/\alpha_{i,i+1}$ are their respective pitches. Thus

$$\mathbf{Z}(\theta_i, d_i) = \mathrm{e}^{\theta_i \mathbf{K}} \quad \text{and} \quad \mathbf{X}(\alpha_{i,i+1}, a_{i,i+1}) = e^{\alpha_{i,i+1} \mathbf{I}} \tag{7}$$

and the kinematics equations (1) become

$$\mathbf{D}(\boldsymbol{q}) = \mathbf{G}\mathrm{e}^{\theta_1 \mathbf{K}} \mathrm{e}^{\alpha_{12} \mathbf{I}} \mathrm{e}^{\theta_2 \mathbf{K}} \ldots \mathrm{e}^{\alpha_{n-1,n} \mathbf{I}} \mathrm{e}^{\theta_n \mathbf{K}} \mathbf{H}, \tag{8}$$

where $\boldsymbol{q} = (\theta_1, \theta_2, \ldots, \theta_n)$ is the joint parameter vector. This is one way to write the product of exponentials form of the kinematics equations. In the next section, this is modified slightly for use as the design equations.

3.2 Relative displacements

If a reference position is chosen for the end effector, denoted by $\mathbf{D}_0$, then the associated joint angle vector $\boldsymbol{q}_0$ can be determined, as well as the world frame coordinates of each of the joint axes. The transformation $\mathbf{D}_0$ is often selected to be the configuration in which the joint parameters are zero and is called the zero reference position by Gupta [**20**].

The displacement of the serial chain relative to this reference configuration is defined by $\mathbf{D}(\Delta \boldsymbol{q}) = \mathbf{D}(\boldsymbol{q})\mathbf{D}(\boldsymbol{q}_0)^{-1}$ and yields a convenient formulation for the kinematics equations. Assume that $\mathbf{D}_0$ is a general position of the end effector defined by joint parameters $\boldsymbol{q}_0$, so $\Delta \boldsymbol{q} = \boldsymbol{q} - \boldsymbol{q}_0$. Then, using the kinematics equations (1)

$$\begin{aligned} \mathbf{D}(\Delta \boldsymbol{q}) = {} & (\mathbf{G}\mathbf{Z}(\theta_1, d_1) \ldots \mathbf{Z}(\theta_n, d_n)\mathbf{H}) \\ & \times (\mathbf{G}\mathbf{Z}(\theta_{10}, d_{10}) \ldots \mathbf{Z}(\theta_{n0}, d_{n0})\mathbf{H})^{-1} \end{aligned} \tag{9}$$

To expand this equation, the following partial displacements are introduced:

$$\mathbf{A}_{i0} = \mathbf{G}\mathbf{Z}(\theta_{10}, d_{10})\mathbf{X}(\alpha_{12}, a_{12}) \ldots \mathbf{X}(\alpha_{i-1,i}, a_{i-1,i}) \tag{10}$$

where, for example

$$\mathbf{A}_{10} = \mathbf{G} \quad \text{and} \quad \mathbf{A}_{20} = \mathbf{G}\mathbf{Z}(\theta_{10}, d_{10})\mathbf{X}(\alpha_{12}, a_{12})$$

Now, insert the identity $\mathbf{Z}(\theta_{i,0})^{-1}\mathbf{A}_{i0}^{-1}\mathbf{A}_{i0}\mathbf{Z}(\theta_{i0}) = \mathbf{I}$ after the first $n-1$ joint transforms $\mathbf{Z}(\theta_i, d_i)$ in equation (9) to obtain the sequence of terms

$$\begin{aligned} \mathbf{T}(\Delta\theta_i, \mathrm{S}_i) &= \mathbf{A}_{i0}\mathbf{Z}(\theta_i, d_i)\mathbf{Z}(\theta_{i,0}, d_{i,0})^{-1}\mathbf{A}_{i0}^{-1} \\ &= \mathbf{A}_{i0}\mathbf{Z}(\Delta\theta_i, \Delta d_i)\mathbf{A}_{i0}^{-1} \end{aligned} \tag{11}$$

The result is the relative transformation that takes the form

$$\mathbf{D}(\Delta \boldsymbol{q}) = \mathbf{T}(\Delta\theta_1, \mathrm{S}_1)\mathbf{T}(\Delta\theta_2, \mathrm{S}_2) \ldots \mathbf{T}(\Delta\theta_n, \mathrm{S}_n) \tag{12}$$

where S_i are the Plucker coordinates of each joint axis obtained by transforming the joint screw K to the world frame by the coordinate transformations defined in equation (11).

Using the exponential form of the transformations $\mathbf{T}(\Delta\theta_i, \mathrm{S}_i)$, the relative kinematics equations (12) are written as

$$\mathbf{D}(\Delta \boldsymbol{q}) = \mathrm{e}^{\Delta\theta_1 \mathbf{S}_1} \mathrm{e}^{\Delta\theta_2 \mathbf{S}_2} \ldots \mathrm{e}^{\Delta\theta_n \mathbf{S}_n} \tag{13}$$

where the matrices $\mathbf{S}_i$ are defined as

$$\mathbf{S}_i = \mathbf{A}_{i0}\mathbf{K}\mathbf{A}_{i0}^{-1} \tag{14}$$

The product of exponential form of the kinematics equations (8) is now obtained as

$$\mathbf{D} = \mathbf{D}(\Delta \boldsymbol{q})\mathbf{D}_0 = \mathrm{e}^{\Delta\theta_1 \mathsf{S}_1}\mathrm{e}^{\Delta\theta_2 \mathsf{S}_2}\ldots\mathrm{e}^{\Delta\theta_n \mathsf{S}_n}\mathbf{D}_0 \tag{15}$$

The difference between this equation and equation (8) is that the coordinates of the joint axes of the serial chain are defined in the world frame.

4 THE EVEN CLIFFORD ALGEBRA $C^+(P^3)$

The Clifford algebra of the projective three space P^3 is a 16-dimensional vector space with a product operation that is defined in terms of a scalar product [**18**]. The elements of even rank form an eight-dimensional subalgebra $C^+(P^3)$ that can be identified with the set of 4×4 homogeneous transforms. Mullineux [**21**] describes the use of this Clifford algebra for motion interpolation, and Daniilidis [**22**], Bayro-Corrochano *et al.* [**23**], and Perez and McCarthy [**24**], describe its use in robot design and camera calibration.

The typical element of $C^+(P^3)$ can be written as the eight-dimensional vector given by

$$\hat{A} = a_0 + a_1 i + a_2 j + a_3 k + a_4\epsilon + a_5 i\epsilon + a_6 j\epsilon + a_7 k\epsilon, \tag{16}$$

where the basis elements i, j, and k are the well-known quaternion units and ϵ is called the dual unit. The quaternion units satisfy the multiplication relations

$$i^2 = j^2 = k^2 = -1, \quad ij = k, \quad jk = i, \quad ki = j, \quad \text{and} \quad ijk = -1 \tag{17}$$

The dual number ϵ commutes with i, j, and k and multiplies by the rule $\epsilon^2 = 0$.

In these calculations, it is convenient to consider the linear combination of quaternion units to be a vector in three dimensions, so the notation $\boldsymbol{A} = a_1 i + a_2 j + a_3 k$ and $\boldsymbol{A}^\circ = a_5 i + a_6 j + a_7 k$ is used (the small circle in the superscript is often used to distinguish coefficients of the dual unit). This allows to write the Clifford algebra element (16) as

$$\hat{A} = a_0 + \boldsymbol{A} + a_4\epsilon + \boldsymbol{A}^\circ\epsilon \tag{18}$$

Now, collecting the scalar and vector terms, this element takes the form

$$\hat{A} = (a_0 + a_4\epsilon) + (\boldsymbol{A} + \boldsymbol{A}^\circ\epsilon) = \hat{a} + \mathsf{A} \tag{19}$$

The dual vector $\mathsf{A} = \boldsymbol{A} + \boldsymbol{A}^\circ\epsilon$ can be identified with the pairs of vectors that define lines and screws [**13**].

Using this notation, the Clifford algebra product of elements $\hat{A} = \hat{a} + \mathsf{A}$ and $\hat{B} = \hat{b} + \mathsf{B}$ takes the form

$$\begin{aligned}\hat{C} &= (\hat{b} + \mathsf{B})(\hat{a} + \mathsf{A}) \\ &= (\hat{b}\hat{a} - \mathsf{B}\cdot\mathsf{A}) + (\hat{a}\mathsf{B} + \hat{b}\mathsf{A} + \mathsf{B}\times\mathsf{A})\end{aligned} \tag{20}$$

where the usual vector dot and cross products are extended linearly to dual vectors.

4.1 Exponential of a vector

The product operation in the Clifford algebra allows to compute the exponential of a vector $\theta\boldsymbol{S}$, where $|\boldsymbol{S}| = 1$, as

$$e^{\theta\boldsymbol{S}} = 1 + \theta\boldsymbol{S} + \frac{\theta^2}{2}\boldsymbol{S}^2 + \frac{\theta^3}{3!}\boldsymbol{S}^3 + \cdots \tag{21}$$

From equation (20), $\boldsymbol{S} = 0 + \boldsymbol{S}$ and compute

$$\begin{aligned}&\boldsymbol{S}^2 = (0 + \boldsymbol{S})(0 + \boldsymbol{S}) = -1, \\ &\boldsymbol{S}^3 = -\boldsymbol{S}, \quad \boldsymbol{S}^4 = 1, \quad \text{and} \quad \boldsymbol{S}^5 = \boldsymbol{S},\end{aligned} \tag{22}$$

which means

$$\begin{aligned}e^{\theta\boldsymbol{S}} &= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \cdots\right) + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)\boldsymbol{S} \\ &= \cos\theta + \sin\theta\boldsymbol{S}\end{aligned} \tag{23}$$

This is the well-known unit quaternion that represents a rotation around the axis $\boldsymbol{S}$ by the angle $\phi = 2\theta$. The rotation angle ϕ is double that given in the quaternion, because the Clifford algebra form of a rotation requires multiplication by both $Q = \cos\theta + \sin\theta\boldsymbol{S}$ and its conjugate $Q^* = \cos\theta - \sin\theta\boldsymbol{S}$. In particular, if $\boldsymbol{x}$ and $\boldsymbol{X}$ are the coordinates of a point before and after the rotation, then the quaternion coordinate transformation equation

$$\boldsymbol{X} = Q\boldsymbol{x}Q^* \tag{24}$$

For this reason, the quaternion is often written in terms of one-half the rotation angle, i.e. $Q = \cos\phi/2 + \sin\phi/2\boldsymbol{S}$.

4.2 Exponential of a screw

The Plücker coordinates $\mathsf{S} = (\boldsymbol{S}, \boldsymbol{C}\times\boldsymbol{S})$ of a line can be identified with the Clifford algebra element $\mathsf{S} = \boldsymbol{S} + \epsilon\boldsymbol{C}\times\boldsymbol{S}$. Similarly, the screw $\mathsf{J} = (\boldsymbol{S}, \boldsymbol{V}) = (\boldsymbol{S}, \boldsymbol{C}\times\boldsymbol{S} + \mu\boldsymbol{S})$ becomes the element $\mathsf{J} = \boldsymbol{S} + \epsilon\boldsymbol{V} =$

$(1+\mu\epsilon)$S. Using the Clifford product, the exponential of the screw θJ is computed,

$$\mathrm{e}^{\theta \mathrm{J}} = 1 + \mathrm{J} + \frac{\theta^2}{2}\mathrm{J}^2 + \frac{\theta^3}{3!}\mathrm{J}^3 + \cdots \tag{25}$$

Note that $\mathrm{S}^2 = -1$, therefore

$$\begin{aligned}&\mathrm{J}^2 = -(1+\mu\epsilon)^2 = -(1+2\mu\epsilon), \quad \mathrm{J}^3 = -(1+3\mu\epsilon)\mathrm{S}\\ &\mathrm{J}^4 = 1 + 4\mu\epsilon \quad \text{and} \quad \mathrm{J}^5 = (1+5\mu\epsilon)\mathrm{S}\end{aligned} \tag{26}$$

yielding

$$\begin{aligned}\mathrm{e}^{\theta J} &= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \cdots\right) + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)\mathrm{S}\\ &\quad - \theta\mu\epsilon\left(\theta - \frac{\theta^3}{3!} + \cdots\right) + \theta\mu\epsilon\left(1 - \frac{\theta^2}{2} + \cdots\right)\mathrm{S}\\ &= (\cos\theta - d\sin\theta\epsilon) + (\sin\theta + d\cos\theta\epsilon)\mathrm{S}\end{aligned} \tag{27}$$

where $d = \theta\mu$ is the slide along the screw axis associated with the angle θ. At this point, it is convenient to introduce the dual angle $\hat{\theta} = \theta + d\epsilon$, so the identities

$$\begin{aligned}&\sin\hat{\theta} = \sin\theta + d\cos\theta\epsilon \quad \text{and}\\ &\cos\hat{\theta} = \cos\theta - d\sin\theta\epsilon\end{aligned} \tag{28}$$

are derived using the series expansions of sine and cosine.

Equation (27) introduces the unit dual quaterion, which is identified with spatial displacements. To see the relationship, the rotation term is factored out to obtain

$$\hat{Q} = \cos\hat{\theta} + \sin\hat{\theta}\mathrm{S} = (1 + \boldsymbol{t}\epsilon)(\cos\theta + \sin\theta\boldsymbol{S}) \tag{29}$$

where

$$\boldsymbol{t} = d\boldsymbol{S} + \sin\theta\cos\theta\,\boldsymbol{C}\times\boldsymbol{S} - \sin^2\theta(\boldsymbol{C}\times\boldsymbol{S})\times\boldsymbol{S} \tag{30}$$

This vector is one-half the translation $\boldsymbol{d} = 2\boldsymbol{t}$ of the spatial displacement associated with this dual quaternion, similar to the relation of the rotation angle $\phi = 2\theta$. This is because the Clifford algebra form of the transformation of line coordinates x to X by the rotation ϕ around an axis S with the translation $\boldsymbol{d}$ involves multiplication by both the Clifford algebra element $\hat{Q} = \cos\hat{\theta} + \sin\hat{\theta}\mathrm{S}$ and its conjugate $\hat{Q}^* = \cos\hat{\theta} - \sin\hat{\theta}\mathrm{S}$, given by

$$\mathrm{X} = \hat{Q}\,\mathrm{x}\,\hat{Q}^* \tag{31}$$

For this reason, the unit dual quaternion is usually written in terms of the half rotation angle and half displacement vector

$$\begin{aligned}\hat{Q} &= \cos\frac{\hat{\phi}}{2} + \sin\frac{\hat{\phi}}{2}\mathrm{S}\\ &= \left(1 + \frac{1}{2}\boldsymbol{d}\epsilon\right)\left(\cos\frac{\phi}{2} + \sin\frac{\phi}{2}\boldsymbol{S}\right)\end{aligned} \tag{32}$$

where

$$\boldsymbol{d} = 2\left(\frac{k}{2}\boldsymbol{S} + \sin\frac{\phi}{2}\cos\frac{\phi}{2}(\boldsymbol{C}\times\boldsymbol{S}) - \sin^2\frac{\phi}{2}(\boldsymbol{C}\times\boldsymbol{S}\times\boldsymbol{S})\right) \tag{33}$$

The dual angle $\hat{\phi} = \phi + k\epsilon$ was obtained by introducing the slide along S given by $k = \phi\mu$.

4.3 Clifford algebra kinematics equations

The exponential of a screw defines a relative displacement from an initial position to a final position in terms of a rotation around and slide along an axis. This means that the composition of Clifford algebra elements defines the relative kinematics equations for a serial chain that are equivalent to equation (13).

Consider the nC serial chain in which each joint can rotate an angle θ_i around, and slide the distance d_i along the axis S_i for $i = 1, \ldots, n$. Let $\boldsymbol{\theta}_0$ and $\boldsymbol{d}_0$ be the joint parameters of this chain in the reference configuration, so

$$\begin{aligned}\Delta\hat{\boldsymbol{q}} &= (\boldsymbol{\theta} + \boldsymbol{d}\epsilon) - (\boldsymbol{\theta}_0 + \boldsymbol{d}_0\epsilon)\\ &= (\Delta\hat{\theta}_1, \Delta\hat{\theta}_2, \ldots, \Delta\hat{\theta}_n)\end{aligned} \tag{34}$$

Then, the movement from this reference configuration is defined by the kinematics equations

$$\begin{aligned}\hat{D}(\Delta\hat{\boldsymbol{q}}) &= \mathrm{e}^{\frac{\Delta\hat{\theta}_1}{2}\mathrm{S}_1}\mathrm{e}^{\frac{\Delta\hat{\theta}_2}{2}\mathrm{S}_2}\cdots\mathrm{e}^{\frac{\Delta\hat{\theta}_n}{2}\mathrm{S}_n}\\ &= \left(c\frac{\Delta\hat{\theta}_1}{2} + s\frac{\Delta\hat{\theta}_1}{2}\mathrm{S}_1\right)\left(c\frac{\Delta\hat{\theta}_2}{2} + s\frac{\Delta\hat{\theta}_2}{2}\mathrm{S}_2\right)\cdots\\ &\quad\times\left(c\frac{\Delta\hat{\theta}_n}{2} + s\frac{\Delta\hat{\theta}_n}{2}\mathrm{S}_n\right)\end{aligned} \tag{35}$$

Note that s and c denote the sine and cosine functions, respectively.

5 DESIGN EQUATIONS FOR A SERIAL CHAIN

The goal of the design problem is to determine the dimensions of a spatial serial chain that can position a tool held by its end effector in a given set of task positions. The location of the base of the robot, the position of the tool frame, and the link dimensions and joint angles are considered to be design variables.

5.1 Specified task positions

Identify a set of task positions $\mathbf{P}_j, j = 1, \ldots, m$. Then, the physical dimensions of the chain are defined by the requirement that for each position $\mathbf{P}_j$, there is a joint parameter vector $\boldsymbol{q}_j$ such that the kinematics equations of the chain satisfy the relations

$$\mathbf{P}_j = \mathbf{D}(\boldsymbol{q}_j), \quad i = 1, \ldots, m \tag{36}$$

Now, choose $\mathbf{P}_1$ as the reference position and compute the relative displacements $\mathbf{P}_j\mathbf{P}_1^{-1} = \mathbf{P}_{1j}, j = 2, \ldots, m$.

For each of these relative displacements $\mathbf{P}_{1j}$, the dual unit quaternion is $\hat{P}_{1j} = \cos(\Delta\hat{\phi}_{1j}/2) + \sin(\Delta\hat{\phi}_{1j}/2)\mathrm{P}_{1j}, j = 2, \ldots, m$. The dual angle $\Delta\hat{\phi}_{1j}$ defines the rotation about and slide along the axis P_{1j}, which defines the displacement from the first to the jth position. Now writing equation (35) for the $m - 1$ relative displacements, the following equation is obtained

$$\hat{P}_{1j} = \mathrm{e}^{\frac{\Delta\hat{\theta}_{1j}}{2}\mathrm{S}_1}\mathrm{e}^{\frac{\Delta\hat{\theta}_{2j}}{2}\mathrm{S}_2}\cdots\mathrm{e}^{\frac{\Delta\hat{\theta}_{nj}}{2}\mathrm{S}_n}, \quad j = 2, \ldots, m \tag{37}$$

The result is $8(m - 1)$ design equations. The unknowns are the n joint axes $\mathrm{S}_i, i = 1, \ldots, n$, and the $n(m - 1)$ pairs of joint parameters $\Delta\hat{\theta}_{ij} = \Delta\theta_{ij} + \Delta d_{ij}\epsilon$.

5.2 Counting

The eight components of the unit Clifford algebra kinematics equations (37) are not independent. In particular, it is easy to see that a dual unit quaternion satisfies the identity

$$\hat{Q}\hat{Q}^* = \mathrm{e}^{\frac{\Delta\hat{\phi}}{2}\mathrm{S}}\mathrm{e}^{\frac{-\Delta\hat{\phi}}{2}\mathrm{S}} = 1 \tag{38}$$

which imposes a pair of constraints: one on the real part and one on the dual part. Only six of the eight design equations for each of the $m - 1$ relative positions are independent, which means there are $6(m - 1)$ design equations.

This is true only for a unit dual quaternion; it can be shown that for a unit dual quaternion defined as a composition of screw rotations, the unit condition is implied by the axes being lines, that is, by each axis satisfying the two Plücker constraints presented in equation (4). This is first shown for a quaternion defined by a single screw displacement, $\hat{Q} = \mathrm{e}^{(\Delta\phi/2)\mathrm{S}}$. If the unit condition is tested, then

$$\begin{aligned}\hat{Q}\hat{Q}^* &= \left(c\frac{\Delta\hat{\phi}}{2} + s\frac{\Delta\hat{\phi}}{2}\mathrm{S}\right)\left(c\frac{\Delta\hat{\phi}}{2} - s\frac{\Delta\hat{\phi}}{2}\mathrm{S}\right)\\ &= c\frac{\Delta\hat{\phi}}{2}c\frac{\Delta\hat{\phi}}{2} + s\frac{\Delta\hat{\phi}}{2}s\frac{\Delta\hat{\phi}}{2}\mathrm{S}\cdot\mathrm{S}\end{aligned} \tag{39}$$

If $\mathrm{S}\cdot\mathrm{S} = 1$, then

$$\begin{aligned}\hat{Q}\hat{Q}^* &= c\frac{\Delta\hat{\phi}}{2}c\frac{\Delta\hat{\phi}}{2} + s\frac{\Delta\hat{\phi}}{2}s\frac{\Delta\hat{\phi}}{2}\\ &= \left(c\frac{\Delta\phi}{2}\right)^2 + \left(s\frac{\Delta\phi}{2}\right)^2 = 1\end{aligned} \tag{40}$$

In general, for a dual quaternion obtained as the composition of transformations about n joint axes,

$$\begin{aligned}\hat{Q}\hat{Q}^* &= \left(\mathrm{e}^{\frac{\Delta\phi_1}{2}\mathrm{S}_1}\cdots\mathrm{e}^{\frac{\Delta\phi_n}{2}\mathrm{S}_n}\right)\\ &\quad\left(\mathrm{e}^{\frac{\Delta\phi_1}{2}\mathrm{S}_1}\cdots\mathrm{e}^{\frac{\Delta\phi_n}{2}\mathrm{S}_n}\right)^*\end{aligned} \tag{41}$$

If this product is expanded, then

$$\begin{aligned}\hat{Q}\hat{Q}^* &= \left(\mathrm{e}^{\frac{\Delta\phi_1}{2}\mathrm{S}_1}\cdots\mathrm{e}^{\frac{\Delta\phi_n}{2}\mathrm{S}_n}\right)\left(\mathrm{e}^{\frac{-\Delta\phi_n}{2}\mathrm{S}_n}\cdots\mathrm{e}^{\frac{-\Delta\phi_1}{2}\mathrm{S}_1}\right)\\ &= \mathrm{e}^{\frac{\Delta\phi_1}{2}\mathrm{S}_1}\cdots\left(\mathrm{e}^{\frac{\Delta\phi_n}{2}\mathrm{S}_n}\mathrm{e}^{\frac{-\Delta\phi_n}{2}\mathrm{S}_n}\right)\cdots\\ &\quad\times\mathrm{e}^{\frac{-\Delta\phi_1}{2}\mathrm{S}_1}\end{aligned} \tag{42}$$

using the associative property of the Clifford algebra product. Applying the previous result, $\mathrm{e}^{(\Delta\phi_n/2)\mathrm{S}_n}\mathrm{e}^{(-\Delta\phi_n/2)\mathrm{S}_n} = 1$ iff $\mathrm{S}_n\cdot\mathrm{S}_n = 1$. Repeat the process for every pair of individual transformations to obtain

$$\hat{Q}\hat{Q}^* = 1 \Leftrightarrow \mathrm{S}_i\cdot\mathrm{S}_i = 1, \quad i = 1, \ldots, n \tag{43}$$

The set of six independent equations in the dual quaternion equalities and the Plücker conditions for each joint axis define the miminum set of independent equations for the design problem.

The joint axis parameters in a chain that consists of r revolute joints and p prismatic joints are counted. For synthesis purposes, a purely prismatic joint is defined by the unit vector $\boldsymbol{S}$ that defines the slide direction, so it has two independent parameters. Any location of the prismatic joint will give the same relative motion at the end effector, the only difference being in the physical structure with respect to the adjacent joints. The revolute joint axis is defined by the Plücker coordinate vectors,

$\mathsf{S} = \boldsymbol{S} + \boldsymbol{C} \times \boldsymbol{S}\epsilon$, that have four independent components due to the conditions in equation (4).

Thus, the joint axes that define this chain have $K = 6r + 3p$ components minus $2r + p$ Plücker constraints, i.e. $4r + 2p$ independent unknowns.

Revolute and prismatic joints each have a single joint parameter, either a rotation angle or a slide distance, which means that the chain has $(r + p)(m - 1)$ unknown joint parameters that define the $m - 1$ relative positions.

Subtracting the number of equations from the number of unknowns yields

$$\begin{aligned} E &= 4r + 2p + (r + p)(m - 1) - 6(m - 1) \\ &= (3r + p + 6) + (r + p - 6)m \end{aligned} \tag{44}$$

where E is the excess of unknowns over equations. This excess can be made equal to zero for chains with dofs $= r + p \leq 5$, in which case

$$m = \frac{3r + p + 6 - c}{6 - (r + p)} \tag{45}$$

task positions are specified. If fewer than this number of task positions are defined, or if the chain has six or more dofs, then the values for the excess design parameters can be freely selected. In equation (45), c has been added to denote any extra constraint that may be imposed on the axes. Table 2 presents the maximum number of positions that can be defined for some chains with five degrees of freedom.

It is interesting to note that because the composition of displacements has the structure of a semi-direct product, the rotations are obtained by operating rotations only. A specific counting scheme can be generated for the rotations by considering only the first quaternion of the dual quaternion. The maximum number of task rotations is obtained as

$$m_R = \frac{3 + r}{3 - r} \tag{46}$$

In some cases with $r = 1$ or 2, the rotation part of the design equations can be used to determine the directions of these axes independently. These chains are called 'orientation limited'. Refer to Perez and McCarthy [**24**] for a discussion of this case.

Table 2 The number of task positions that determine the structural parameters for five-dof serial chains

Chain	K	Task positions	Total equations
PRPRP	21	15	91
RPRPR	24	17	104
RRRRP	27	19	117
RRRRR	30	21	130

5.3 Special cases: T, S, and PP joints

The counting formula in equation (45) is used for revolute and prismatic joints assembled in serial chains. The RR and RRR chains can be further specialized by introducing geometric constraints between their joint axes to define the universal and spherical joints, respectively. In addition, two consecutive prismatic joints span the group of displacements T_P, planar translations on plane P, and they form a special type of joint called PP. The following subsections show how in some of these cases, the number of design parameters is less than that obtained by considering directly the geometric constraints on the axes.

5.3.1 The T joint

Consider the RR chain formed by axes S_i and S_{i+1}. If these axes are required to intersect at a right angle, then a Hooke's joint is obtained, also called a universal joint, which is denoted by a T following Crane and Duffy [**25**]. This geometric constraint is defined by the dual vector equation

$$\mathrm{T}: \quad \mathsf{S}_i \cdot \mathsf{S}_{i+1} = 0 \tag{47}$$

which expands to define the two constraints

$$\mathrm{T}: \quad \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} = 0 \quad \text{and} \quad \boldsymbol{S}_i \cdot \boldsymbol{S}^\circ_{i+1} + \boldsymbol{S}^\circ_i \cdot \boldsymbol{S}_{i+1} = 0 \tag{48}$$

The design equations for the RRR chain, for instance, are easily transformed into design equations for the TR chain by including these two constraint equations with the appropriate indices.

5.3.2 The S joint

In the same way, a sequence of three revolute joints, RRR chain, can be constrained such that they intersect at a point, and the pairs in sequence are perpendicular. This is a common construction for an active spherical joint, denoted by S, which allows full orientation freedom around the intersection point. However, for synthesis applications, it can be shown that any three axes create the same shperical joint.

Denote three axes as S_i, S_{i+1}, and S_{i+2}, then the equations that define this joint consist of the dual vector constraints

$$\mathrm{S}: \quad \mathrm{S}_i \cdot \mathrm{S}_{i+1} = 0, \quad \mathrm{S}_i \cdot \mathrm{S}_{i+2} = 0, \quad \text{and} \quad \mathrm{S}_{i+1} \cdot \mathrm{S}_{i+2} = 0 \tag{49}$$

If the spherical joint is written as the dual quaternion product of these individual axes

$$\hat{S}(\theta_1, \theta_2, \theta_3) = \hat{S}_1(\theta_1)\hat{S}_2(\theta_2)\hat{S}_3(\theta_3) \tag{50}$$

when expanded, it gives

$$\hat{S}(\theta_1, \theta_2, \theta_3) = \alpha_4 + \alpha_1 \mathrm{S}_1 + \alpha_2 \mathrm{S}_2 + \alpha_3 \mathrm{S}_3 \tag{51}$$

where each α_i appears as combinations of the joint variables

$$\begin{aligned}
\alpha_1 &= \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cos\frac{\theta_3}{2} + \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_3}{2} \\
\alpha_2 &= \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_3}{2} - \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin\frac{\theta_3}{2} \\
\alpha_3 &= \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_3}{2} + \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin\frac{\theta_3}{2} \\
\alpha_4 &= \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cos\frac{\theta_3}{2} - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_3}{2}
\end{aligned} \tag{52}$$

Now we show how any directions $\boldsymbol{S}_1$, $\boldsymbol{S}_2$, $\boldsymbol{S}_3$ can be used to define the spherical joint. Equate equation (50) to a goal displacement $\hat{P} = (p_w + \epsilon p_w^0) + (\boldsymbol{P} + \epsilon \boldsymbol{P}^0)$

$$\hat{S}(\theta_1, \theta_2, \theta_3) = \hat{P} \tag{53}$$

and solve linearly for the combinations of joint variables in the α_i factors using the real part of the dual quaternion equation

$$\begin{bmatrix} \boldsymbol{S}_1 & \boldsymbol{S}_2 & \boldsymbol{S}_3 & \boldsymbol{0} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{P} \\ p_w \end{Bmatrix} \tag{54}$$

where the scalar term is placed in the fourth row. The values obtained for the joint angles,

$$\alpha_1 = \boldsymbol{S}_1 \cdot \boldsymbol{P}, \quad \alpha_2 = \boldsymbol{S}_2 \cdot \boldsymbol{P}, \quad \alpha_3 = \boldsymbol{S}_3 \cdot \boldsymbol{P}, \quad \alpha_4 = p_w \tag{55}$$

are related by the following expression

$$\mathcal{R}: \quad (\boldsymbol{S}_1 \cdot \boldsymbol{P})^2 + (\boldsymbol{S}_2 \cdot \boldsymbol{P})^2 + (\boldsymbol{S}_3 \cdot \boldsymbol{P})^2 + p_w^2 = 1 \tag{56}$$

Noticing that $1 - p_w^2 = \boldsymbol{P} \cdot \boldsymbol{P}$, this expression can be written as

$$\mathcal{R}': \quad (\boldsymbol{S}_1 \cdot \boldsymbol{P})\boldsymbol{S}_1 + (\boldsymbol{S}_2 \cdot \boldsymbol{P})\boldsymbol{S}_2 + (\boldsymbol{S}_3 \cdot \boldsymbol{P})\boldsymbol{S}_3 = \boldsymbol{P} \tag{57}$$

which states that the vector sum of the projections of $\boldsymbol{P}$ on the three joint directions is equal to $\boldsymbol{P}$. This equation holds for any three perpendicular directions.

The expressions of the joint variables are substituted in the dual part of equation (53). Owing to the fact that the last component of the dual quaternion in equation (51) is equal to zero, a spherical joint cannot perform the most general relative displacement.

If the dual part of each joint axis is expressed as $\boldsymbol{S}_i^0 = \boldsymbol{C} \times \boldsymbol{S}_i$, where $\boldsymbol{C}$ is the common intersection point, the dual part of the equations becomes

$$\mathcal{M}: \quad (\boldsymbol{S}_1 \cdot \boldsymbol{P})\boldsymbol{C} \times \boldsymbol{S}_1 + (\boldsymbol{S}_2 \cdot \mathbf{P})\boldsymbol{C} \times \boldsymbol{S}_2 + (\boldsymbol{S}_3 \cdot \boldsymbol{P})\boldsymbol{C} \times \boldsymbol{S}_3 = \boldsymbol{P}^0 \tag{58}$$

and this is equal to

$$\mathcal{M}': \quad \boldsymbol{C} \times ((\boldsymbol{S}_1 \cdot \boldsymbol{P})\boldsymbol{S}_1 + (\boldsymbol{S}_2 \cdot \boldsymbol{P})\boldsymbol{S}_2 + (\boldsymbol{S}_3 \cdot \boldsymbol{P})\boldsymbol{S}_3) = \boldsymbol{P}^0 \tag{59}$$

Observe that the expression in parenthesis is the left-hand side of equation (57). Use this to obtain

$$\mathcal{M}'': \quad \boldsymbol{C} \times \boldsymbol{P} = \boldsymbol{P}^0 \tag{60}$$

This set of three equations specifies two out of the three coordinates of the point. At least two relative positions need to be defined to fully specify this point. However, as noted previously, they cannot be general positions, if an exact solution is required.

Thus for a spherical joint, only the coordinates of the intersection point $\boldsymbol{C}$ and the three joint angles are design variables. This gives a different counting than the one obtained by solving for three perpendicular and intersecting revolute joints.

5.3.3 The PP joint

When a serial robot is to be designed with two consecutive prismatic joints, these can be made to be coplanar, as the location of a prismatic joint is not a design parameter. The set of displacements produced by the two prismatic joints forms the subgroup T_P of planar translations on a plane P. The subgroup has dimension 2, and two more parameters are needed to define the direction normal to the plane; for synthesis purposes, the location of the plane is again arbitrary.

The PP joint is created by using two prismatic joints, which is a total of four joint parameters plus two joint slides; however, both directions do not appear independently in the equations. Let $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ be the directions of the two prismatic joints. The displacements of the PP joint are obtained as the dual quaternion product

$$\hat{S}(d_1, d_2) = \hat{S}_1(d_1)\hat{S}_2(d_2) \tag{61}$$

When expanded, it yields

$$\hat{S}^0(d_1, d_2) = 1 + \epsilon\left(\frac{d_1}{2}\boldsymbol{S}_1 + \frac{d_2}{2}\boldsymbol{S}_2\right) \tag{62}$$

The joint variables d_1 and d_2 are solved in the dual part of the design equations

$$\begin{bmatrix} \frac{1}{2}\boldsymbol{S}_1 & \frac{1}{2}\boldsymbol{S}_2 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \boldsymbol{P}^0 \tag{63}$$

For the system to have a solution, the determinant of the augmented matrix must be zero, this yields the simplified design equation

$$\mathcal{M}: \quad (\boldsymbol{S}_1 \times \boldsymbol{S}_2) \cdot \boldsymbol{P}^0 = 0 \tag{64}$$

The parameters of the prismatic joints always appear as a cross product, and it can be substituted by the common normal, $\boldsymbol{S}_1 \times \boldsymbol{S}_2 = \boldsymbol{N}$. Within the plane defined by $\boldsymbol{N}$, any two independent directions can be used to define the joint axes.

Thus, for synthesis purposes, the design parameters for two consecutive prismatic joints are the two slides and the vector $\boldsymbol{N}$ defining the normal direction to $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$. It coincides with the parameters needed to define the subgroup T_P.

Other cases in which the number of design parameters is less than that obtained by imposing extra constraints on the joint axes can be found in a similar way.

6 ASSEMBLING THE DESIGN EQUATIONS

The structure of the Clifford algebra design equations provides a systematic approach to assemble the design equations for a broad range of serial chains. The basic approach is to formulate the design equations for the nC serial chain, and then restrict the joint variables to form prismatic or sliding joints and impose geometric conditions on the axes to form universal or spherical joints or to account for specific geometry. The result is a systematic way of defining the design equations for a broad range of chains. The procedure for the 3C serial chain is presented, and it has been implemented in a numerical solver, as well as for the 2C, 4C, and 5C cases.

6.1 The 3C chain

The Clifford algebra form of the relative kinematics equations for the 3C chain can be written as

$$\hat{D}(\Delta\hat{\boldsymbol{\theta}}) = \left(c\frac{\Delta\hat{\theta}_1}{2} + s\frac{\Delta\hat{\theta}_1}{2}\mathrm{S}_1\right)\left(c\frac{\Delta\hat{\theta}_2}{2} + s\frac{\Delta\hat{\theta}_2}{2}\mathrm{S}_2\right) \times \left(c\frac{\Delta\hat{\theta}_3}{2} + s\frac{\Delta\hat{\theta}_3}{2}\mathrm{S}_3\right) \tag{65}$$

where $\mathrm{S}_i = \boldsymbol{S}_i + \boldsymbol{S}_i^\circ\epsilon$ defines the ith joint axis in the reference position and $\Delta\hat{\theta}_i = \Delta\theta_i + \Delta d_i$ defines the rotation and slide of the cylindric joint around the ith axis.

Expand the right-hand side of equation (65) using the Clifford product to obtain

$$\begin{aligned} \hat{D}(\Delta\hat{\boldsymbol{\theta}}) &= (\hat{c}_1\hat{c}_2 - \hat{s}_1\hat{s}_2\mathrm{S}_1 \cdot \mathrm{S}_2 + \hat{s}_1\hat{c}_2\mathrm{S}_1 \\ &\quad + \hat{c}_1\hat{s}_2\mathrm{S}_2 + \hat{s}_1\hat{s}_2\mathrm{S}_1 \times \mathrm{S}_2)(\hat{c}_3 + \hat{s}_3\mathrm{S}_3), \\ &= \hat{c}_1\hat{c}_2\hat{c}_3 - \hat{s}_1\hat{s}_2\hat{c}_3\mathrm{S}_1 \cdot \mathrm{S}_2 - \hat{s}_1\hat{c}_2\hat{s}_3\mathrm{S}_1 \cdot \mathrm{S}_3 \\ &\quad - \hat{c}_1\hat{s}_2\hat{s}_3\mathrm{S}_2 \cdot \mathrm{S}_3 - \hat{s}_1\hat{s}_2\hat{s}_3\mathrm{S}_1 \times \mathrm{S}_2 \cdot \mathrm{S}_3 \\ &\quad + \hat{s}_1\hat{c}_2\hat{c}_3\mathrm{S}_1 + \hat{c}_1\hat{s}_2\hat{c}_3\mathrm{S}_2 + \hat{c}_1\hat{c}_2\hat{s}_3\mathrm{S}_3 \\ &\quad + \hat{s}_1\hat{s}_2\hat{c}_3\mathrm{S}_1 \times \mathrm{S}_2 + \hat{s}_1\hat{c}_2\hat{s}_3\mathrm{S}_1 \times \mathrm{S}_3 \\ &\quad + \hat{c}_1\hat{s}_2\hat{s}_3\mathrm{S}_2 \times \mathrm{S}_3 + \hat{s}_1\hat{s}_2\hat{s}_3 \\ &\quad \times ((\mathrm{S}_1 \times \mathrm{S}_2) \times \mathrm{S}_3 - (\mathrm{S}_1 \cdot \mathrm{S}_2)\mathrm{S}_3) \end{aligned} \tag{66}$$

For convenience, notations $\hat{c}_i = \cos\Delta\hat{\theta}_i/2$ and $\hat{s}_i = \sin\Delta\hat{\theta}_i/2$ are introduced.

Equation (66) can be written in matrix form to emphasize that it is the linear combination of the eight monomials formed as products of the joint angles, which is assembled into an array in reversed lexicographic order,

$$\begin{aligned} \hat{\boldsymbol{V}} = \big(&\hat{c}_1\hat{c}_2\hat{c}_3,\ \hat{s}_1\hat{c}_2\hat{c}_3,\ \hat{c}_1\hat{s}_2\hat{c}_3,\ \hat{c}_1\hat{c}_2\hat{s}_3, \\ &\hat{s}_1\hat{s}_2\hat{c}_3,\ \hat{s}_1\hat{c}_2\hat{s}_3,\ \hat{c}_1\hat{s}_2\hat{s}_3,\ \hat{s}_1\hat{s}_2\hat{s}_3\big)^{\mathrm{T}} \end{aligned} \tag{67}$$

To do this, the vector form of the dual unit quaternion $\hat{\boldsymbol{Q}} = \cos\Delta\hat{\theta}/2 + \sin(\Delta\hat{\theta}/2)\mathrm{S}$, is introduced, given by

$$\hat{Q} = \begin{Bmatrix} \sin\frac{\Delta\hat{\theta}}{2}(S_x + S_x^{\circ}\epsilon) \\ \sin\frac{\Delta\hat{\theta}}{2}(S_y + S_y^{\circ}\epsilon) \\ \sin\frac{\Delta\hat{\theta}}{2}(S_z + S_z^{\circ}\epsilon) \\ \cos\frac{\Delta\hat{\theta}}{2} \end{Bmatrix} = \begin{Bmatrix} \sin\frac{\Delta\hat{\theta}}{2}\mathsf{S} \\ \cos\frac{\Delta\hat{\theta}}{2} \end{Bmatrix} \tag{68}$$

Collecting terms in equation (66), the following matrix is obtained

$$\hat{\mathbf{D}}(\Delta\hat{\boldsymbol{\theta}}) = \begin{bmatrix} 0 & \mathsf{S}_1 & \mathsf{S}_2 & \mathsf{S}_3 & \mathsf{S}_1\times\mathsf{S}_1 & \mathsf{S}_1\times\mathsf{S}_3 & \mathsf{S}_2\times\mathsf{S}_3 & -(\mathsf{S}_1\cdot\mathsf{S}_2)\mathsf{S}_3 + (\mathsf{S}_1\times\mathsf{S}_2)\times\mathsf{S}_3 \\ 1 & 0 & 0 & 0 & -\mathsf{S}_1\cdot\mathsf{S}_2 & -\mathsf{S}_1\cdot\mathsf{S}_3 & -\mathsf{S}_2\cdot\mathsf{S}_3 & -\mathsf{S}_1\times\mathsf{S}_2\cdot\mathsf{S}_3 \end{bmatrix}\hat{\boldsymbol{V}} \tag{69}$$

The Clifford algebra notation is compact in that each column of this matrix actually forms a column of four dual coefficients, or eight real coefficients if the dual components of the dual quaternion are written after the real components, forming an eight-dimensional vector. Similarly, each of the monomials in $\hat{\mathbf{V}}$ expands into four real terms, which are listed as

$$\boldsymbol{M} = \left(\boldsymbol{V}, \frac{\Delta d_1}{2}\boldsymbol{V}, \frac{\Delta d_2}{2}\boldsymbol{V}, \frac{\Delta d_3}{2}\boldsymbol{V}\right) \tag{70}$$

where $\boldsymbol{V}$ is the array of real parts of $\hat{\boldsymbol{V}}$. Thus, equation (69) expands to an 8×32 matrix equation. The number k of joint variable monomials in an nC serial chain is given by

$$k = (n+1)2^n \tag{71}$$

Thus, these equations become 8×12 for 2C, 8×80 for 4C, and 8×192 for 5C chains.

The kinematics equations (69) can be used directly for the design of a 3C chain. In what follows, these equations are specialized to obtain the design equations for a variety of special serial chains.

6.2 RCC, RRC, and RRR chains

The ith cylindric joint in the 3C chain is converted to a revolute joint simply by setting $\Delta d_i = 0$. This can be done in seven different ways to define three permutations of the RRC chain, three permutations of the RCC chain, and the RRR chain.

For example, the monomials in equation (69) that define the RCC, CRC, or CCR chains are given by

$$\begin{aligned} &\text{RCC:} \quad \boldsymbol{M} = \left(\boldsymbol{V}, \frac{\Delta d_2}{2}\boldsymbol{V}, \frac{\Delta d_3}{2}\boldsymbol{V}\right) \\ &\text{CRC:} \quad \boldsymbol{M} = \left(\boldsymbol{V}, \frac{\Delta d_1}{2}\boldsymbol{V}, \frac{\Delta d_3}{2}\boldsymbol{V}\right) \\ &\text{CCR:} \quad \boldsymbol{M} = \left(\boldsymbol{V}, \frac{\Delta d_1}{2}\boldsymbol{V}, \frac{\Delta d_2}{2}\boldsymbol{V}\right) \end{aligned} \tag{72}$$

Similarly, the RRC, RCR, and CRR chains have the monomials

$$\begin{aligned} &\text{RRC:} \quad \boldsymbol{M} = \left(\boldsymbol{V}, \frac{\Delta d_3}{2}\boldsymbol{V}\right) \\ &\text{RCR:} \quad \boldsymbol{M} = \left(\boldsymbol{V}, \frac{\Delta d_2}{2}\boldsymbol{V}\right) \\ &\text{CRR:} \quad \boldsymbol{M} = \left(\boldsymbol{V}, \frac{\Delta d_1}{2}\boldsymbol{V}\right) \end{aligned} \tag{73}$$

Finally, the RRR chain is defined by the monomial list

$$\text{RRR:} \quad \boldsymbol{M} = \boldsymbol{V} \tag{74}$$

Note that if an nC chain is specialized to have r revolute joints, then the number of monomials is given by

$$k = (n - r + 1)2^n \tag{75}$$

6.3 PCC, PPC, and PPP chains

A two-step process is required to convert the ith cylindric joint to a prismatic joint. The first step is to set $\Delta\theta_i = 0$. The second step consists of specializing the joint axis $\mathsf{S}_i = \mathbf{S}_i$, so that its dual part is zero. This latter constraint arises because the pure translation defined by a prismatic joint depends only on the direction, not on the location in space of its axis.

To define the monomials for the three permutations of the PCC chain, $\mathbf{W}_1 = (c_1c_2c_3, c_1s_2c_3, c_1c_2s_3, c_1s_2s_3)$ is introduced, and $\mathbf{W}_2$ and $\mathbf{W}_3$, where the subscript i indicates that s_i is made equal to zero, are similarly defined. This allows to define the arrays of monomials

$$\begin{aligned} &\text{PCC:} \quad \boldsymbol{M} = \left(\mathbf{W}_1, \frac{\Delta d_1}{2}\mathbf{W}_1, \frac{\Delta d_2}{2}\mathbf{W}_1, \frac{\Delta d_3}{2}\mathbf{W}_1\right) \\ &\text{CPC:} \quad \boldsymbol{M} = \left(\mathbf{W}_2, \frac{\Delta d_1}{2}\mathbf{W}_2, \frac{\Delta d_2}{2}\mathbf{W}_2, \frac{\Delta d_3}{2}\mathbf{W}_2\right) \\ &\text{CCP:} \quad \boldsymbol{M} = \left(\mathbf{W}_3, \frac{\Delta d_1}{2}\mathbf{W}_3, \frac{\Delta d_2}{2}\mathbf{W}_3, \frac{\Delta d_3}{2}\mathbf{W}_3\right) \end{aligned} \tag{76}$$

The monomials for the three permuations of the PPC chain are easily determined by introducing the set of monomials $\mathbf{W}_{12} = (c_1c_2c_3, c_1c_2s_3)$ and similarly $\mathbf{W}_{13}$ and $\mathbf{W}_{23}$

$$\begin{aligned}
\text{PPC:}\quad \boldsymbol{M} &= \left(\mathbf{W}_{12}, \frac{\Delta d_1}{2}\mathbf{W}_{12}, \frac{\Delta d_2}{2}\mathbf{W}_{12}, \frac{\Delta d_3}{2}\mathbf{W}_{12}\right) \\
\text{PCP:}\quad \boldsymbol{M} &= \left(\mathbf{W}_{13}, \frac{\Delta d_1}{2}\mathbf{W}_{13}, \frac{\Delta d_2}{2}\mathbf{W}_{13}, \frac{\Delta d_3}{2}\mathbf{W}_{13}\right) \\
\text{CPP:}\quad \boldsymbol{M} &= \left(\mathbf{W}_{23}, \frac{\Delta d_1}{2}\mathbf{W}_{23}, \frac{\Delta d_2}{2}\mathbf{W}_{23}, \frac{\Delta d_3}{2}\mathbf{W}_{23}\right)
\end{aligned} \tag{77}$$

Finally, the PPP chain is defined by the monomial list

$$\begin{aligned}
\text{PPP:}\quad \boldsymbol{M} = \Bigg(&(c_1c_2c_3), \frac{\Delta d_1}{2}(c_1c_2c_3), \\
&\frac{\Delta d_2}{2}(c_1c_2c_3), \frac{\Delta d_3}{2}(c_1c_2c_3)\Bigg)
\end{aligned} \tag{78}$$

The number of monomials in an nC chain with p of the joints restricted to be prismatic is seen to be

$$k = (n+1)2^{n-p} \tag{79}$$

Table 3 summarizes the constraints needed to transform the C joint into the most common types of joints. Notice that, for the spherical joint and other special cases, the approach of adding constraints between consecutive joint axes is used. This will not yield the minimum set of joint parameters, but it gives satisfactory results with the numerical solver.

This approach to the formulation of the design equations for special cases of the CCC chain can be extended to any nC chain.

7 SYNTHESIS OF 5C AND RELATED CHAINS

In this section, a numerical synthesis algorithm is presented which uses the Clifford algebra exponential design equations for the 5C serial chain (Fig. 2).

Table 3 Constraints that specialize C joints to R, P, T, and S joints

Joint	Axes	Constraints
R	S_i	$\Delta d_i = 0$
P	S_i	$\Delta\theta_i = 0$
C	S_i	None
T	S_i, S_{i+1}	$\Delta d_i = 0, \Delta d_{i+1} = 0, S_i \cdot S_{i+1} = 0$
S	S_i, S_{i+1}, S_{i+2}	$\Delta d_i = 0, \Delta d_{i+1} = 0, \Delta d_{i+2} = 0$ $S_i \cdot S_{i+1} = 0, S_{i+1} \cdot S_{i+2} = 0, S_i \cdot S_{i+2} = 0$

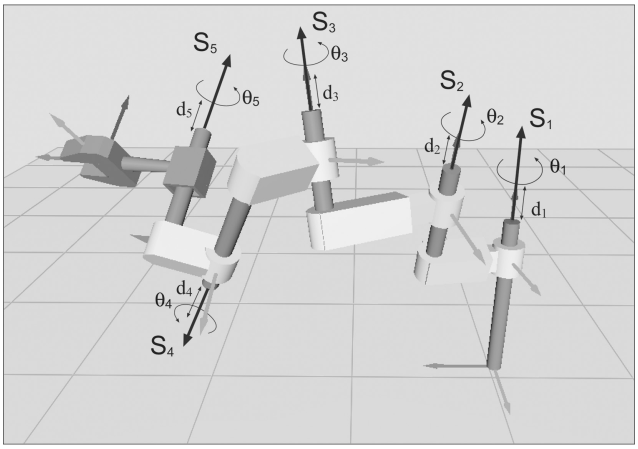

Fig. 2 The 5C serial robot

The special cases of this chain include robots with up to five joints and up to ten degrees of freedom.

The design equations for a specific serial robot are to be obtained from the 5C robot equations by imposing conditions on some of the axes or joint variables. The kinematics equations for the 5C robot are given by

$$\begin{aligned}
\hat{Q}_{5C} &= e^{\frac{\Delta\hat{\theta}_1}{2}S_1} e^{\frac{\Delta\hat{\theta}_2}{2}S_2} e^{\frac{\Delta\hat{\theta}_3}{2}S_3} e^{\frac{\Delta\hat{\theta}_4}{2}S_4} e^{\frac{\Delta\hat{\theta}_5}{2}S_5} \\
&= \left(\cos\frac{\Delta\hat{\theta}_1}{2} + \sin\frac{\Delta\hat{\theta}_1}{2}S_1\right)\left(\cos\frac{\Delta\hat{\theta}_2}{2} + \sin\frac{\Delta\hat{\theta}_2}{2}S_2\right) \\
&\quad \cdots \left(\cos\frac{\Delta\hat{\theta}_5}{2} + \sin\frac{\Delta\hat{\theta}_5}{2}S_5\right)
\end{aligned} \tag{80}$$

The kinematics equations for a serial chain consisting of revolute (R), prismatic (P), universal (T), cylindrical (C), or spherical (S) joints can be obtained from the 5C robot using the approach presented in the previous section. For example, the kinematics equation of the TPR robot is obtained by requiring the axes S_1 and S_2 to be perpendicular and coincident, setting the joint variables d_1, d_2, θ_3, and d_4 to zero. The extra joint is eliminated by setting θ_5 and d_5 to zero. Other joints, such as the helical (H) or planar (E) joints, can also be modelled by imposing constraints on the axes and joint parameters.

To facilitate the specialization of the general 5C robot to a specific serial chain topology, its kinematics equations are organized as a linear combination of the products of joint angles and slides, which are considered to be monomials with coefficients defined by the structural parameters of the joints. Structured in this way, the kinematics equations of the 5C robot are a linear combination of 192 monomials. These monomials can be organized into six sets of 32 products of sines and cosines of the $\Delta\theta_i$ joint angles, which can be assembled into the vector

$$V = (s_1s_2s_3s_4s_5, (s_1s_2s_3s_4c_5)_5, (s_1s_2s_3c_4c_5)_{10}, (s_1s_2c_3c_4c_5)_{10}, (s_1c_2c_3c_4c_5)_5, c_1c_2c_3c_4c_5) \quad (81)$$

where $c_i = \cos(\Delta\theta_i/2)$, $s_i = \sin(\Delta\theta_i/2)$. The parentheses $()_j$ denote the j possible permutations of each set of sines and cosines. The remaining five sets of monomials are obtained by multiplying $\boldsymbol{V}$ by the joint slides $\Delta d_i/2$, hence a total set of monomials $\boldsymbol{M}$ is expressed as

$$\boldsymbol{M} = \left(\boldsymbol{V}, \frac{\Delta d_1}{2}\boldsymbol{V}, \frac{\Delta d_2}{2}\boldsymbol{V}, \frac{\Delta d_3}{2}\boldsymbol{V}, \frac{\Delta d_4}{2}\boldsymbol{V}, \frac{\Delta d_5}{2}\boldsymbol{V}\right) \quad (82)$$

The kinematics equations of the 5C robot can now be written as the linear combination

$$\hat{Q}_{5C} = \sum_{i=1}^{192} \boldsymbol{K}_i m_i, \quad m_i \in \boldsymbol{M} \quad (83)$$

The coefficients $\boldsymbol{K}_i$ are eight-dimensional vectors containing the structural variables defining the joint axes.

This equation is adjusted to accommodate a revolute or prismatic joint inserted as the jth joint axis by selecting the non-zero components of the vector $\boldsymbol{M}$. Note that, if the jth C joint is replaced by a revolute joint, then the slide Δd_j is zero, which eliminates 32 components in $\boldsymbol{M}$. Similarly, if this joint is replaced by a prismatic joint, then the angle $\Delta\theta_j = 0$, which eliminates 16 terms from the vector $\boldsymbol{V}$.

These equations are constructed starting with the array $L_{5C} = \{1, 2, \ldots, 192\}$ of indices that denote the components of $\boldsymbol{M}$ for the general 5C chain, sorted as indicated above. The arrays L_{R_j}, L_{P_j}, and L_{C_j} denote the non-zero components of $\boldsymbol{M}$ for the cases when joint j is a revolute, prismatic, or cylindrical joint, respectively. These arrays are given by

$$\begin{aligned} L_{R_j} &= \left\{i : \left(\cos\frac{\Delta\theta_j}{2} \wedge \sin\frac{\Delta\theta_j}{2}\right) \in m_i \vee \frac{\Delta d_j}{2} \notin m_i\right\} \\ L_{P_j} &= \left\{i : \left(\frac{\Delta d_j}{2} \wedge \cos\frac{\Delta\theta_j}{2}\right) \in m_i \vee \sin\frac{\Delta\theta_j}{2} \notin m_i\right\} \\ L_{C_j} &= \left\{i : \left(\frac{\Delta d_j}{2} \wedge \cos\frac{\Delta\theta_j}{2} \wedge \sin\frac{\Delta\theta_j}{2}\right) \in m_i\right\} \end{aligned} \quad (84)$$

where $\wedge$ and $\vee$ denote the logical OR, and AND operations, respectively. Finally, the array of indices L for a specific serial chain topology is computed by intersecting the arrays obtained for all of the joints, that is

$$L = \bigcap_{j=1}^{5} (L_{R_j} \cup L_{P_j} \cup L_{C_j}) \quad (85)$$

where $L_{P_j} = \emptyset$ and $L_{C_j} = \emptyset$ if j is a revolute joint, for example.

The kinematics equations for the specific serial chain is now given by

$$\hat{Q} = \sum_{i \in L} \boldsymbol{K}_i m_i \quad (86)$$

The synthesis equations for the chain are obtained by equating the kinematics equations in equation (86) to the set of task positions $\hat{P}_{1i}$, that is

$$\hat{Q} = \hat{P}_{1i}, \quad i = 2, \ldots, m \quad (87)$$

where the maximum number of task positions, m, is obtained for the chosen topology using equations (45) and (46). Additional constraint equations may be added to account for the specialized geometry of T and S joints or for any other geometric constraint present in the robot.

These synthesis equations are solved to determine the joint axes S_i in the reference configuration, as well as for values for the joint variables that ensure that the serial chain reaches each of the task positions.

7.1 Synthesis algorithm

An algorithm for the formulation and numerical solution of the synthesis equations for serial chains has been implemented in the java-based design software Synthetica 2.0 [**26**]. This algorithm creates specific topologies by specializing the general 5C chain. For convenience, implementation separates the general case into the four subclasses of 2C, 3C, 4C, and 5C related serial chains.

The synthesis equations are solved numerically using a Java translation by Steve Verrill of the Levenberg–Marquardt algorithm in the FORTRAN MINPACK source produced by Garbow, Hillstrom, and others. The input data consist of the set of task positions and the topology of the serial chain. The topology of the chains is used to construct its kinematics equations $\hat{Q}$. These equations are set equal to the task positions $\hat{P}_{1i}$ to yield the synthesis equations as the difference, $\hat{Q} - \hat{P}_{1i}$, $i = 2, \ldots, m$. The numerical solver finds values for the components of the joint axes and joint variables that minimize this difference.

Table 4 Task of relative positions to synthesize serial chains

Position	Dual quaternion
1	Identity
2	((0.02535, −0.1474, 0.5806, 0.8003), (0.6600, −0.2973, 0.08989, −0.1409))
3	((0.06318, −0.3675, 0.3791, 0.8469), (0.7705, −0.3797, 0.1974, −0.3106))
4	((−0.02824, −0.4374, −0.8982, 0.03244), (−0.3837, 0.7690, −0.3383, 0.6676))
5	((0.4115, −0.2907, −0.4150, 0.7576), (0.3204, 0.3672, 0.1397, 0.04343))
6	((−0.04529, −0.4268, −0.9020, 0.04803), (−0.3938, 0.7421, −0.2971, 0.6427))
7	((0.2846, −0.01418, −0.3655, 0.8861), (−0.06626, −0.06411, 0.1066, 0.06422))
8	((0.3629, 0.3513, 0.4772, 0.7191), (−0.4320, −0.5377, −0.1164, 0.5579))
9	((−0.1819, 0.7938, 0.02589, 0.5798), (−0.7315, −0.9360, −0.5138, 1.075))
10	((−0.04342, 0.9348, −0.2038, 0.2875), (−1.015, −0.6633, −0.9378, 1.339))
11	((0.07687, 0.9269, −0.3672, 0.007037), (−1.111, −0.3781, −1.161, 1.365))
12	((−0.1883, −0.8085, 0.4857, 0.2737), (1.076, 0.07761, 1.235, −1.223))
13	((−0.2919, −0.5372, 0.5513, 0.5677), (0.8691, −0.2563, 1.106, −0.8693))
14	((−0.4481, −0.5161, 0.5141, 0.5183), (0.7849, −0.2161, 1.019, −0.5474))
15	((−0.6824, −0.4463, 0.4176, 0.4009), (0.5950, −0.1318, 0.8025, 0.03009))
16	((−0.8586, −0.3212, 0.3032, 0.2603), (0.3796, −0.05823, 0.4581, 0.6466))
17	((−0.9260, −0.1912, 0.2636, 0.1908), (0.2702, −0.07531, 0.1021, 1.095))
18	((−0.8879, −0.08849, 0.3694, 0.2595), (0.3217, −0.2424, −0.2018, 1.305))
19	((−0.7381, −0.002776, 0.5437,0.3994), (0.3954, −0.4890, −0.4246, 1.305))
20	((−0.5203, 0.05977, 0.6776, 0.5163), (0.4024, −0.6879, −0.5098, 1.154))
21	((−0.3710, 0.09429, 0.7306, 0.5655), (0.3714, −0.7775, −0.5147, 1.038))

The implementation of the numerical solver does not attempt to solve the minimum set of design equations. The minimum number of equations is defined by equation (45). Instead, all $8(m-1)+c$ synthesis equations are used in the numerical solver. For the cases of 3R, 4R, and 5R serial chains, this approach introduces two, eight, and 30 redundant equations, respectively. Experience shows that these additional equations enhance the convergence of the numerical algorithm. Previous experience using the minimum set of independent design equations led to a much longer computational time, duplication of solutions due to the ambiguity in the sign of the joint directions, and many false solutions that were degenerate cases. It seems that the extra equations add a redundancy

Table 5 The positions used to design serial chain robots and the computation time

Chain	dof	Position	Positions selected	Time
TC	4	6	(5, 9, 13, 17, 21)	1.8 s
RRC	4	7	(2, 5, 9, 13, 17, 21)	1.7 s
TRP	4	7	(2, 5, 9, 13, 17, 21)	46.5 s
RRRP	4	8	(2, 3, 5, 9, 13, 17, 21)	7.4 s
TT	4	7	(2, 5, 9, 13, 17, 21)	1.8 min
TRR	4	8	(2, 3, 5, 9, 13, 17, 21)	54.0 s
RRRR	4	9	(2, 3, 4, 5, 9, 13, 17, 21)	17.7 s
SF	5	6	(5, 9, 13, 17, 21)	19.9 s
PSP	5	8	(2, 3, 5, 9, 13, 17, 21)	37.6 s
RCC	5	13	(2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 17, 21)	25.2 s
RRPC	5	14	(2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 17, 21)	17.4 s
RPRC	5	15	(2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 17, 21)	22.3 s
TPC	5	12	(2, 3, 4, 5, 6, 7, 8, 9, 13, 17, 21)	69.7 s
PTC	5	13	(2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 17, 21)	1.7 min
TRF	5	13	(2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 17, 21)	1.6 min
TPRP	5	15	(2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 17, 21)	2.7 min
RRRF	5	15	(2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 17, 21)	3.3 min
RRPRP	5	17	(2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 21)	7.2 min
SC	5	8	(2, 3, 5, 9, 13, 17, 21)	16.9 s
SRP	5	10	(2, 3, 4, 5, 6, 9, 13, 17, 21)	3.0 min
TRC	5	15	(2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 17, 21)	2.6 min
RRRC	5	17	(2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 21)	2.3 min
TTP	5	15	(2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 17, 21)	5.3 min
TRRP	5	17	(2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 21)	3.6 min
RRRRP	5	19	(2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21)	1.3 min
ST	5	10	(2, 3, 4, 5, 6, 9, 13, 17, 21)	27.6 s
SRR	5	12	(2, 3, 4, 5, 6, 7, 8, 9, 13, 17, 21)	5.2 min
TTR	5	17	(2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 21)	1.7 min
TRRR	5	19	(2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21)	2.8 min
RRRRR	5	21	(2–21)	12.6 min

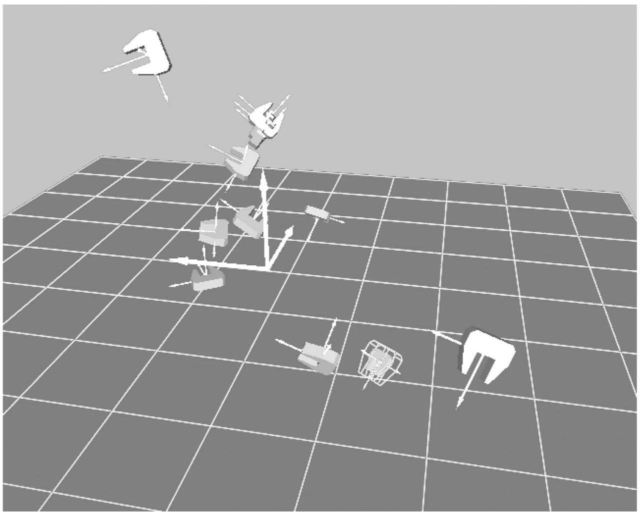

Fig. 3 The 12 positions defining the task

that is useful to eliminate these degenerate cases and to enhance the speed of convergence.

A random start value is generated to initialize the numerical solver. The solver restarts with a new random value if convergence is not achieved within the specified limit. Although this approach cannot guarantee convergence to a solution, so far the solver has been able to always find a solution within a reasonable time; experiments show that the solver regularly returns solutions using from zero to four restarts.

Examples of the computation time for a variety of chains are presented in Table 5. These chains are synthesized with a task of 21 randomly generated positions. From these, the 20 relative dual quaternions indicated in Table 4 are computed. The positions used for each serial chain are indicated in Table 5.

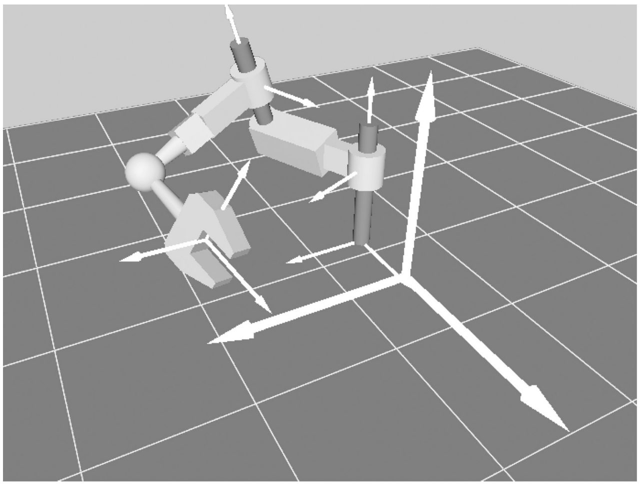

Fig. 4 The CCS serial robot

Table 6 Task positions and values for the first joint angles

Task	Dual quaternion coordinates
1	(0.02, −0.15, 0.58, 0.80, 0.66, −0.30, 0.09, −0.14)
2	(0.06, −0.37, 0.38, 0.85, 0.77, −0.38, 0.20, −0.31)
3	(−0.03, −0.44, −0.90, 0.03, −0.38, 0.77, −0.34, 0.67)
4	(0.41, −0.29, −0.41, 0.76, 0.32, 0.37, 0.14, 0.04)
5	(−0.04, −0.43, −0.90, 0.05, −0.39, 0.74, −0.30, 0.64)
6	(0.28, −0.01, −0.36, 0.89, −0.07, −0.06, 0.11, 0.06)
7	(0.36, 0.35, 0.48, 0.72, −0.43, −0.54, −0.12, 0.56)
8	(−0.18, 0.79, 0.03, 0.58, −0.73, −0.94, −0.51, 1.07)
9	(−0.29, −0.54, 0.55, 0.57, 0.87, −0.26, 1.106, −0.87)
10	(−0.93, −0.19, 0.26, 0.19, 0.27, −0.07, 0.10, 1.09)
11	(−0.37, 0.09, 0.73, 0.56, 0.37, −0.78, −0.51, 1.04)
Joint variables	Values
θ_1	$i\pi/11$, $i = 1, \ldots, 11$
d_1	$0.2i$, $i = 1, \ldots, 11$

8 APPLICATION EXAMPLE

To demonstrate the kinematic synthesis of spatial serial chains, a goal trajectory was determined using the Bezier interpolation of a set of spatial key frames. From this trajectory, 12 positions shown in Fig. 3 are selected to define the task. The topology of the chain is specified to be the seven-dof CCS serial robot. Note that this is a special case of a 5C serial chain, obtained by requiring the last three C joints to be restricted to revolute joints with axes that intersect in a single point.

The CCS robot consists of a shoulder and an elbow that allow both rotation and translation about and along the axes, combined with a spherical wrist (Fig. 4). In addition, the joint angles of the shoulder C joint are constrained to have specific values for the rotation and translation in each of the task positions (see Table 6 for the task positions and the angle specification).

For this example, the Java software was run twice to obtain two different solutions. The first solution took two iterations of the solver with a total time of 91 s. The second solution took 61 s and one iteration. Refer to Table 7 for the coordinates of the joint axes. Figure 5 shows one of the resulting robots moving along the desired task.

Table 7 CCS robots designed to perform the specified task

Solution	Joint axis	Value
1	1	(0.14, 0.49, 0.86, −3.59, −0.45, 0.85)
	2	(−0.24, 0.92, 0.29, −2.44, −0.73, 0.26)
	Wrist	(0.23, 1.21, −0.12)
2	1	(0.72, − 0.50, 0.48, 0.40, 0.03, −0.57)
	2	(−0.10, 0.52, −0.85, −2.53, 0.17, 0.41)
	Wrist	(0.05, 0.72, −0.94)

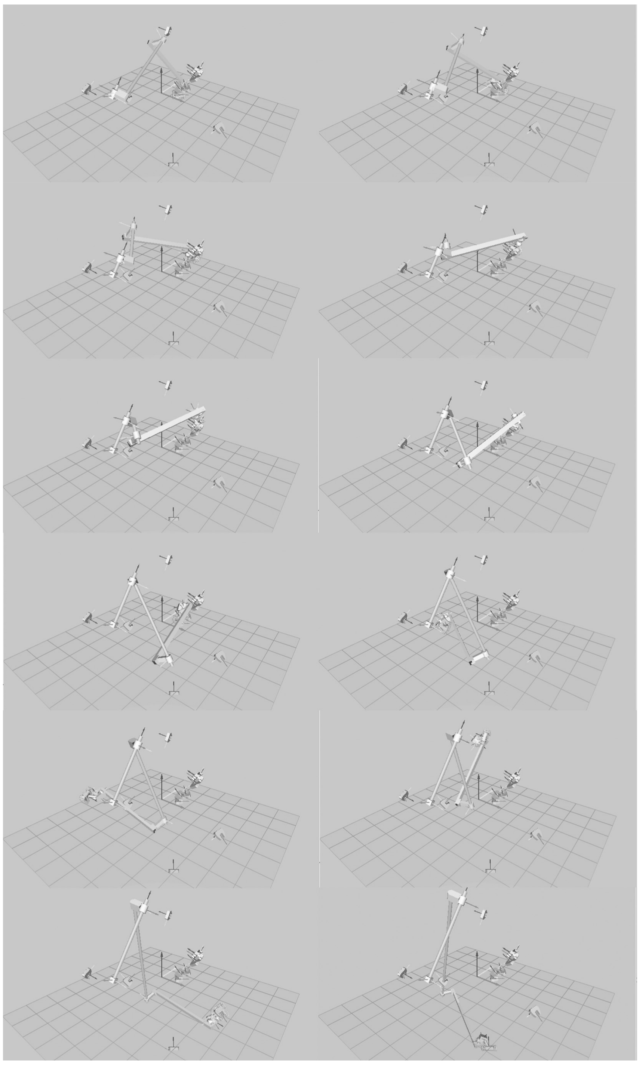

Fig. 5 A solution CCS robot (61 s and one run)

9 SUMMARY

This article uses the kinematics equations of a spatial nC chain to formulate the generalized inverse kinematic problem. Rather than simply compute the joint angles of a robot for a given task trajectory, the structural parameters are also sought for. This is a design problem with applications to modular and reconfigurable robotic systems. This approach may also be useful in the calibration of robots.

The exponential form of the kinematics equations of the chain is reformulated using Clifford algebra to obtain an efficient and systematic set of equations. The derivation of the design equations for robots derived from 2C to 5C chains is automated and verified using a numerical solver.

These generalized inverse kinematic problems become complicated rapidly. In particular, fitting a 5R chain to a 21 task trajectory requires the solution of 130 equations in 130 unknowns. Although individual solutions can be obtained numerically, a bound on the total number of chains that can fit a given task is unknown at this time.

The solution of this problem is demonstrated by determining the structural parameters of a CCS serial chain so that it reaches an arbitrarily specified 12 position task trajectory. In this problem, the values for the first two joint parameters may be freely specified and the remaining parameters are solved for numerically.

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