RL Circuits

Consider the following circuit:

At time \( t = 0 \) the switch is closed. Applying Kirchoff’s loop rule to this circuit and looking at the potentials due to the applied EMF, the resistor and the inductor:

\[
\xi - iR - L \frac{di}{dt} = 0 \rightarrow \frac{di}{dt} = \frac{\xi - iR}{L} \rightarrow \frac{di}{dt} = \frac{\xi}{L} - \frac{R}{L} i
\]

At time \( t = 0, i = 0 \):

\[
\left( \frac{di}{dt} \right)_{i=0} = \frac{\xi}{L}
\]

Notice that a large value of \( L \) has the effect of slowing the rate of change (in this case growth) of current in the circuit.

At any time \( t \neq 0 \), \( \frac{R}{L} i \neq 0 \), the instantaneous current in the circuit is \( i \) and

\[
\frac{di}{dt} = \frac{\xi}{L} - \frac{R}{L} i
\]

- Notice that as the quantity \( \frac{R}{L} i \) increases, \( \frac{di}{dt} \) gets smaller.
- As \( \frac{R}{L} i \) approaches the same numeric value as \( \frac{\xi}{L} \), \( \frac{di}{dt} \rightarrow 0 \)
- As \( t \rightarrow \infty, i = I \) and when \( i = I \), \( \frac{di}{dt} = 0 \) (steady state).
- In the steady state, \( \frac{di}{dt} = 0 = \frac{\xi}{L} - \frac{R}{L} I \rightarrow \frac{\xi}{L} = \frac{R}{L} I \therefore \xi = IR \)
This is in agreement with our knowledge of how resistive circuits behave in the steady state.

- Notice that the final, steady-state value of the current does not depend on the self-inductance of the circuit.
- Only the time it takes to reach the steady-state is affected and this occurs during the transient phase of the circuit’s operation.

We seek a relationship between the current and time during the transient phase of the RL circuit’s operation.

Consider the voltage drops across the inductor and resistor.

\[ V_L = L \frac{di}{dt}, \quad V_R = iR \]

\[ V_{\text{total}} = V_L + V_R = iR + L \frac{di}{dt} = \xi \]

This is another D.E. that we can solve by separation of variables.

\[ \frac{di}{dt} = \frac{V - iR}{L} \Rightarrow \int \frac{di}{V - iR} = \int \frac{dt}{L} \]

\[ L \frac{di}{dt} = V - iR \]

\[ \frac{L \, di}{R \, dt} = \frac{V}{R} - i \]

\[ \frac{di}{V - i} = \frac{R}{L} \, dt \]

\[ \int \frac{di}{V - i} = \int \frac{R}{L} \, dt \]

\[ -\ln \left( \frac{V}{R} - i \right) = \left( \frac{R}{L} \right) t + C \]

Where \( C \) is a combination of both constants of integration.
Now let's look at the initial conditions to evaluate $C$.

At $t = 0$, $i = 0$, so evaluating the previous expression yields:

$$-\ln\left(\frac{V}{R} - i\right) = \left(\frac{R}{L}\right)t \rightarrow -\ln\left(\frac{V}{R}\right) = 0 \therefore C = -\ln\left(\frac{V}{R}\right)$$

$$-\ln\left(\frac{V}{R} - i\right) = \left(\frac{R}{L}\right)t - \ln\left(\frac{V}{R}\right)$$

$$\ln\left(\frac{V}{R} - i\right) - \ln\left(\frac{V}{R}\right) = -\left(\frac{R}{L}\right)t$$

$$\ln\left[\frac{\left(\frac{V}{R} - i\right)}{\frac{V}{R}}\right] = -\left(\frac{R}{L}\right)t$$

$$\ln\left(1 - \frac{iR}{V}\right) = -\left(\frac{R}{L}\right)t$$

$$1 - \frac{iR}{V} = e^{-\left(\frac{R}{L}\right)t}$$

$$\frac{iR}{V} = 1 - e^{-\left(\frac{R}{L}\right)t}$$

$$i = \frac{V}{R}\left(1 - e^{-\left(\frac{R}{L}\right)t}\right)$$

$$i(t) = I\left(1 - e^{-\left(\frac{R}{L}\right)t}\right)$$

This is what we need. We have our time dependant relationship.
Now let’s check this equation at the boundaries of this circuit. At time $t = 0$:

$$i(t) = I \left(1 - e^{-\left(\frac{R}{L}\right)t}\right) \rightarrow i(t) = I(1 - 1) = 0$$

in agreement with our assumptions.

As $t \to \infty$

$$i(t) = I \left(1 - e^{-\left(\frac{R}{L}\right)t}\right) \rightarrow i(t) = I(1 - 0) = I_0$$

also in agreement with our assumptions.

Note that $\frac{R}{L} = \frac{1}{\tau} \cdot \frac{R}{L} = \frac{\tau}{T}$ or $\frac{R}{L} = \frac{1}{\tau} \rightarrow \frac{L}{R} = \tau$

- At time $t = \tau$ the current has risen to $\left(1 - \frac{1}{e}\right)$ or 63% of its final value.
- The time constant for an RL circuit depends on both $L$ and $R$.
- For a given value of $R$, $\tau$, increases with $L$ as expected.
- Current changes more rapidly with smaller values of $L$, also as expected.

What if the RL circuit is operating in the steady state and the switch is opened? Then we would expect current to decrease with time and it is easy to show that:

$$i = I \left(e^{-\left(\frac{R}{L}\right)t}\right) = \frac{\xi}{R} \left(e^{-\left(\frac{R}{\tau}\right)t}\right)$$
The decay of current in a RL circuit

- At time $t = \tau$ the current has decayed to within $\frac{1}{e}$ or 37% of its final value of 0 or down 63% from its initial value, $I$.
- Current changes more rapidly with smaller values of $L$, as expected.

Energy/Power in an RL Circuit

The instantaneous rate at which the source of EMF delivers power to the circuit is:

$$P = Vi$$

The instantaneous rate at which energy is dissipated by the resistor is:

$$P = i^2R$$

The rate at which energy is stored in the inductor is:

$$P = Vi = Li \frac{di}{dt}$$

So the total power in a RL circuit is: $\bar{\tau}i = Li \frac{di}{dt} + i^2R$, where the first term represents the power absorbed by the inductor and the second term represents the power dissipated by the resistor.