ANISOTROPIC BENDING ENERGIES OF CURVES

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ABSTRACT. We study the shape of an elastic rod subject to both bending and twisting, when the rod's resistance to bending depends on the direction of the deformation. In this sense, we develop the theory of anisotropic rods in the plane and in space.

Keywords: anisotropic, elastica, Kirchhoff rod
AMS classification: 53A04, 58E10

1. INTRODUCTION

The study of elasticae links two classical subjects; the theory of curves and the mechanics of solids. They were studied by Galileo, the Bernoulli family, Euler, Kirchhoff, Born and many others. Their study played an important part in the development of elliptic functions and the calculus of variations. For a detailed historical background, we refer the reader to [14]. We give some highlights below.

In 1691, the problem of determining the bending deformation of rods was first formulated by J. Bernoulli (see, for instance, [14, 15, 18]). The simplest version of this classical problem consists on determining the shape of an ideal elastic rod, i.e. a thin elastic rod with circular cross sections and uniform density, naturally straight and prismatic when unstressed and which is being held bent and twisted by external forces and moments acting at its ends alone. Since then, if the rod's resistance to bending is independent of the direction, there have been many results about this topic as well as some nice applications ([2, 6, 7, 8, 9, 11, 12, 13] to mention some).

In particular, if there is no twist and the rod is bent in a plane, so that the center line of the rod becomes a planar curve, we have the Euler-Bernoulli planar elastica. As suggested by D. Bernoulli, nephew of J. Bernoulli, an elastic rod should bend along the curve which minimizes the potential energy of the strain under suitable constraints. Therefore, in 1742, in a letter to L. Euler, he proposed to study elasticae as minimizers of the bending energy.

Using this formulation of elastica as a variational problem, the possible qualitative types for untwisted planar rod configurations were described by L. Euler in his book of 1744, [4], although J. Bernoulli had already partially solved this problem between 1692 and 1694, [1].

Much later, Kirchhoff modeled an elastic rod subject to both bending and twisting, by coupling to a curve in 3-space, the center line, a frame in the orthogonal space and adding the energy of this frame to the elastic energy. In this setting,

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J. Langer and D. Singer [12], formulated a variational problem for the center line alone using an affine combination of the bending energy and the total torsion. In fact, they proved that the center line of a Kirchhoff elastic rod in equilibrium is an extremal curve for this combination of functionals.

However, if the rod's resistance depends on the direction of the bending deformation, less is known about their shape. This resistance, which is measured with the Young's modulus, is in this case anisotropic.

It is to be expected that rods which have some type of internal structures such as fibers, grains or strata might exhibit this characteristic. It may also be the case that this anisotropic response to bending may result from idealizing an object of unequal dimensions, such as long rectangular plank, as a curve. Anisotropic resistance to bending may also be the result of the rod being placed in an anisotropic medium. In some sense, the classical version of the bending energy is already anisotropic since the rod can only bend along or, in the case of the Kirchhoff rod, twist around, its tangent direction. Here however we consider a type of 'extrinsic' anisotropy, where the resistance to bending is governed by the orientation of the curve in the ambient space.

In this paper, we study the geometry of the rod when its resistance to bending depends on the direction of the curves's tangent. To do this, we adopt a device from crystallography which is known as the Wulff construction, [19]. We begin with a suitable positive function $\gamma(\cdot)$ on the S^n, $(n = 1, 2)$. We regard $\gamma(T)$ as measuring the unit energy per unit length of a curve having unit tangent vector T. Thus, γ defines a Minkowski-Finsler metric. The *Wulff shape* of γ is the set

$$W := \partial\{\vec{v} \in \mathbf{R}^{n+1} \mid \vec{v} \cdot T \leqslant \gamma(T),\ \forall T \in S^n\}\,.$$

In this paper, the function γ is restricted by the condition that W, which is always convex, is also smooth. For a sufficiently smooth curve $C \in \mathbf{R}^{n+1}$, the *Cahn-Hoffman* field ξ assigns to each point of C, the point $\xi \in W$ where the unit tangent at C agrees with the unit normal to W. Our definition of this field is an adaption to the theory of curves of a field introduced to study surfaces [3]. We then define the *anisotropic bending energy* of C as the energy of the map ξ. In the isotropic case ($\gamma \equiv 1$) this definition agrees with the usual one. The critical curves for the anisotropic bending energy will be called anisotropic elastica or constrained anisotropic elastica if the arc-length of the curve is constrained to be fixed. The definition of the anisotropic bending energy allows us to generalize the definition of the Kirchhoff rod to the anisotropic case.

There are certainly other models for anisotropic rods in the literature, which we contrast with our's below.

As expected, the equilibrium equations for the elastica and Kirchhoff rods lead to non linear fourth order equations. Our main approach is to reduce the order by studying the Cahn-Hoffman field. Solving for the elastica is basically reduced to studying geodesics in W and solutions of the second order gradient flow for a component of the Gauss map of W, while for the Kirchhoff rod, an additional term containing the geodesic curvature of the Cahn-Hoffman map must be included. We use this approach to prove the existence of a minimizer for the anisotropic bending energy of a curve with clamped endpoints.

Despite the generality involved, these definitions lead to a surprisingly coherent theory. For example, in the planar case, we are able to give integral formulas for elastica. While in the three dimensional axially symmetric case, we can reduce the construction of many anisotropic elasticae to quadratures.

The paper is organized as follows. In Chapter 2, we give a short derivation of the anisotropic bending energy for planar curves. In Chapter 3, we study the anisotropic energy of space curves. Chapter 4 discusses the anisotropic Kirchhoff rod in three dimensional space. In Chapter 5, we discuss results for elastica in three space by specializing the results of the previous chapter. Chapter 6 discusses further specialization to the case of axially symmetric energies.

2. Motivation

Let C denote a smooth, regular planar curve which we regard as representing a bent rod having uniform cross sections. We consider the curve as lying in the xz plane. At a point p on C, we consider an infinitesimal arc of length ds. To second order, we can replace this arc by the corresponding arc of the osculating circle to C at p; that is, the circle having the same curvature and tangent as C has at p. We denote the radius of this circle (the *radius of curvature* of C) by R. Under a displacement of the curve C, this radius undergoes a change $R \to R + x$ and the arc-length undergoes a change $ds = R d\theta \to ds' = (R + x) d\theta$. It follows that the strain is given by

$$\epsilon := \frac{ds' - ds}{ds} = \frac{x}{R},$$

and the corresponding stress is

$$\sigma = E\epsilon = E\frac{x}{R},$$

where E denotes the *Young's modulus*, which measures the rod's resistance to bending. Letting S denote the cross-sectional surface of the rod, it then follows that the contribution to the potential energy of the curve from the infinitesimal arc is given by

$$d\mathcal{E} = \frac{1}{2}\frac{E}{R}\left(\int_S x^2 \, dS\right) \, ds = \frac{1}{2}E\kappa^2 I_y \, ds,$$

where $\kappa = 1/R$ is the curvature of C at p and I_y is the moment of inertia of S about the y axis. (See [10] page 75 or Chapter 38 of [5] for details).

We now postulate that the Young's modulus is anisotropic, i.e. that it depends on the direction of the bending deformation. This means that the material of the rod is more resistant to bending in some directions than in others. We also assume, that this resistance to bending varies smoothly with the direction so that we can represent the modulus as a smooth, positive function $E(\theta)$. We intend to represent this function as the reciprocal of the square of the curvature of a convex curve Ω. In fact, it is easy to see that the necessary and sufficient condition for the existence of Ω is the mild condition

$$\int_0^{2\pi} e^{i\theta}\sqrt{E(\theta)}\,d\theta = 0\,. \tag{1}$$

Proposition 2.1. *Let* $E(\theta)$ *be a real valued continuous function on the circle* S^1 *such that* E *has finite left and right hand derivatives at each point. Then if* E *satisfies (1), there exists a twice differentiable solution of*

$$\gamma_{\theta\theta} + \gamma = \sqrt{E}\,, \tag{2}$$

on S^1.

Any two such solutions differ by a homogeneous trigonometric polynomial of degree 1.

Proof. Under the given hypothesis, we can express E as a Fourier series,

$$\sqrt{E(\theta)} = (z_0 + \bar{z}_0) + \sum_{j\geqslant 2}(z_j e^{ij\theta} + \bar{z}_j e^{-ij\theta})\,.$$

Define

$$\gamma(\theta) := (z_0 + \bar{z}_0) + \sum_{j\geqslant 2}\frac{(z_j e^{ij\theta} + \bar{z}_j e^{-ij\theta})}{1-j^2}\,.$$

Then $\gamma \in C^2(S^1)$ and it is straightforward to see that γ satisfies (2).

The final statement in the theorem follows since $\cos\theta$ and $\sin\theta$ span the kernel if $\gamma \mapsto \gamma'' + \gamma$. **q.e.d.**

In particular, condition (1) will be satisfied, for example, if E is an even function of θ. Setting $\mu(\theta)$ equal to the curvature of Ω, we arrive at the potential energy functional

$$\mathcal{E}[C] = m\int_C \left(\frac{\kappa(s)}{\mu(\theta(s))}\right)^2 ds\,,$$

where $\theta(s)$ is the angle that the tangent to the curve C makes with the positive horizontal axis and m is a positive constant.

To obtain the energy functional given above in a more mathematical manner, we need to begin by considering a first order anisotropic line energy defined for a regular parameterized curve in the plane. We let $\gamma : S^1 \to \mathbf{R}^+$ denote a sufficiently smooth function satisfying the following *convexity condition*. We require that if θ denotes the usual polar angle in the plane, then

$$\frac{1}{\mu} := \gamma_{\theta\theta} + \gamma > 0$$

holds. The function μ represents the curvature of the plane curve given by

$$\chi := \theta \mapsto (\gamma_\theta - i\gamma)e^{i\theta}\,,$$

where we have identified $\mathbf{R}^2$ with the complex plane $\mathbf{C}$. The (convex) curve Ω defined by χ will be referred to as the *Wulff shape*. For purposes that will be clear later, we introduce the curve $\Omega^\perp$ which is a clockwise rotation of Ω through an angle $\pi/2$.

For any smooth, regular curve $C : I \to \mathbf{R}^2$, we denote by T its unit tangent and by N its unit normal with $N = JT$ where J is counter-clockwise rotation by an angle $\pi/2$. We assume that s is the arc-length parameter along C. As above, we represent T by $e^{i\theta}$ and write $\gamma(\theta)$ as $\gamma(T)$ when desired.

We define the energy of C by

$$\mathcal{F}[C] = \int_C \gamma(T)\, ds\, .$$

In the integrand, the function γ should be considered as the unit energy per unit length of an infinitesimal piece of the curve having tangent T and length ds. Note that this is a special type of Finsler metric often referred to as a Minkowski-Finsler functional.

To obtain the equilibrium conditions, we subject the curve C to a variation, i.e. for small values of ϵ we consider a family of curves $C(\epsilon, s) := C(s) + \epsilon(\delta C(s))$, where $\delta C := \phi(s)T + \zeta(s)N$. We do not impose any boundary conditions on the variation yet. If κ denotes the curvature of C, then the *anisotropic curvature* is defined by

$$\lambda(s) := \frac{\kappa(s)}{\mu(\theta(s))}\, . \tag{3}$$

The first variation of the functional $\mathcal{F}$ is then given by

$$\delta\mathcal{F}[C] = -\int_C \lambda N \cdot \delta C\, ds + \left(J\chi\left(\theta(s)\right) \cdot \delta C\, \right|_0^L ,$$

where $I = [0, L]$, L denoting the length of C.

If the endpoints are fixed, then equilibria must satisfy $\lambda \equiv 0$ so the equilibria are just straight lines. If the curves are assumed to be closed and are constrained to enclose a fixed area, then the critical points will satisfy $\lambda \equiv$ constant. The (closed) solutions of this problem are homothetic to $\Omega^\perp$. In fact, *rescalings of $\Omega^\perp$ are the absolute minimizers of $\mathcal{F}$ among all curves enclosing the same area*. This result, which is a special case of *Wulff's Theorem*, is a generalization of the isoperimetric problem in the plane.

As suggested before, we will consider the functionals

$$\mathcal{E}_\beta[C] := \int_C \left(\lambda^2 + \beta\right)\, ds$$

The case $\beta \neq 0$ amounts to constraining the arc-length to be a fixed constant. The case $\beta = 0$ will be called the *unconstrained* case. We regard β as a Lagrange multiplier which fixes the length of the curve representing a flexible, inextensible rod.

Note that if we define the map

$$\xi := \chi \circ T : C \to \Omega\, ,$$

then

$$\mathcal{E}_0[C] = \int_C \|d\xi\|^2\, ds\, ,$$

which is the energy of the map ξ. We will call the map $\xi : C \to \Omega$ the *Cahn-Hoffman field*. Note that the Cahn-Hoffman field of $\Omega^\perp$ is just a counter-clockwise rotation through an angle $\pi/2$.

Since $\xi_s = \theta_s i(\gamma_{\theta\theta} + \gamma)e^{i\theta} = \lambda N$, it follows that the normal line to $C(s)$ and the tangent line to Ω at $\xi(s)$ are parallel. This means that the normal to Ω is the tangent $T = e^{i\theta}$ and

$$\theta_s = \kappa\,, \quad \theta_\sigma = \mu\,,$$

at corresponding points, where σ denotes the arc-length along Ω. From this we get $d\sigma = \lambda\, ds$. If ℓ denotes the length of Ω, we get using the Cauchy-Schwartz inequality,

$$\ell \leqslant \int_C |d\sigma| = \int_C |\lambda|\, ds \leqslant \left(\int_C \lambda^2\, ds\right)^{1/2} \left(\int_C\, ds\right)^{1/2}\,,$$

and, therefore,

$$\frac{\ell^2}{L} \leqslant \mathcal{E}_0\,.$$

If instead, we use the inequality between the geometric and arithmetic means, we get that for any $a \in \mathbf{R}^*$

$$\ell \leqslant \frac{a^2}{2}(\mathcal{E}_0 + \frac{1}{a^4}L)\,.$$

So if $\beta > 0$ holds, we take $\beta = a^{-4}$ and get

$$2\sqrt{\beta}\,\ell \leqslant \mathcal{E}_\beta.$$

Equality can only hold when $\lambda \equiv$ constant, i.e. the curve C is a rescaling of $\Omega^\perp$.

If the curve C is not closed, but the angle θ_i, $i = 1, 2$ that the tangent to the curve makes with the horizontal direction at the endpoints are known, an a priori lower bound for $\mathcal{E}_0$ can still be found. Since the map $\chi : S^1 \to \Omega$ is a bijection, θ_i, $i = 1, 2$ determine two arc of Ω which join the two points of Ω where the normal to Ω makes the angle θ_i with the horizontal. We let $\underline{\sigma}(\theta_1, \theta_2)$ be the shorter of the lengths of the two arcs. We can then use

$$\underline{\sigma}(\theta_1, \theta_2) \leqslant \int_C |d\sigma| = \int_C \left|\frac{d\sigma}{ds}\right| ds = \int_C |\lambda|\, ds \leqslant \left(\int_C \lambda^2\, ds\right)^{1/2} \left(\int_C\, ds\right)^{1/2}\,,$$

so that

$$\frac{(\underline{\sigma}(\theta_1, \theta_2))^2}{L} \leqslant \mathcal{E}_0\,.$$

Also, if $\beta > 0$ holds, then

$$2\sqrt{\beta}\,\underline{\sigma}(\theta_1, \theta_2) \leqslant \mathcal{E}_\beta \tag{4}$$

is an a priori lower bound for the functional.

To obtain the first variation formula, we start by computing the pointwise first variation of the anisotropic curvature with respect to the variation field $\delta C = \phi(s)T + \zeta(s)N$. We get

$$\delta\lambda = \partial_s\left(\frac{\zeta_s}{\mu}\right) + \frac{\kappa^2\zeta}{\mu} + \phi\lambda_s\,.$$

From this and integrating by parts, we get

$$\begin{aligned}
\delta\mathcal{E}_\beta[C] &= \int_C 2\lambda\delta\lambda + \left(\lambda^2+\beta\right)\left(-\kappa\zeta+\phi_s\right)\, ds \\
&= \int_C 2\lambda\left(\partial_s\left[\frac{\zeta_s}{\mu}\right]+\frac{\kappa^2\zeta}{\mu}\right) - \left(\lambda^2+\beta\right)\kappa\zeta\, ds + \left(\left[\lambda^2+\beta\right]\phi\,\right|_0^L \\
&= \int_C 2\zeta(\partial_s\left[\frac{\lambda_s}{\mu}\right]+\frac{1}{2}\kappa[\lambda^2-\beta])\, ds + (2\,\frac{\zeta_s\lambda-\zeta\lambda_s}{\mu}+[\lambda^2+\beta]\phi\big|_0^L.
\end{aligned}$$

Regardless of the boundary conditions, an equilibrium curve must satisfy the equation

$$\partial_s\left(\frac{\lambda_s}{\mu}\right)+\frac{1}{2}\kappa\left(\lambda^2-\beta\right)=0 \tag{5}$$

in its interior. We will refer to such a curve as an *anisotropic elastic curve for the energy density* γ. Note that for any γ, straight lines ($\lambda=0$) are obvious examples. A complete classification of anisotropic elastic curves for symmetric densities can be found in [17].

Remark. Note that any rescaling of $\Omega^\perp$ is a solution of (5). For $r\Omega^\perp$, $0<r\in\mathbf{R}$, we have $\lambda\equiv 1/r$ so $r\Omega^\perp$ is a critical point for $\mathcal{E}_{r^{-2}}$. In addition, if $C\subset r\Omega^\perp$ is the shorter of two arcs connecting distinct points in $r\Omega^\perp$, then C realizes the lower bound (4).

Remark. Assume $\Omega^\perp$ is invariant with respect to reflection across the vertical axis and let W be the surface of revolution obtained by rotating $\Omega^\perp$ about this axis. The convex surface W is the Wulff shape for an anisotropic energy functional defined for surfaces in $\mathbf{R}^3$. When $\beta=0$ a cylinder over an anisotropic elastic curve will be a critical point for the anisotropic Willmore energy corresponding to this functional which is given by

$$S\mapsto\int_S \Lambda^2\, dS\,,$$

where Λ is the *anisotropic mean curvature*. For details see [16].

Assume that λ is not constant. Multiplying by λ_s/μ we can see that equation (5) possesses the first integral

$$\left(\frac{\lambda_s}{\mu}\right)^2+\frac{(\lambda^2-\beta)^2}{4}\equiv \text{constant} =: p^2\,. \tag{6}$$

In fact, the case $\lambda\equiv$constant can also be included considering that p may vanish. That is, $p=0$ corresponds to rescalings of $\Omega^\perp$. In this case $C=r\Omega^\perp$ and we have that $\beta=1/(r^2)$.

If $p\neq 0$ we can set

$$\lambda_s/\mu = p\cos\hat\theta, \qquad (\lambda^2-\beta)/2 = p\sin\hat\theta \tag{7}$$

for some function $\hat\theta(s)$. Differentiating the second equation above, we get

$$\hat\theta_s p\cos\hat\theta=\hat\theta_s\lambda_s/\mu=(p\sin\hat\theta)_s=\lambda\lambda_s\,,$$

so on any interval where λ is not constant, we get $\hat{\theta}_s = \lambda\mu = \kappa = \theta_s$, i.e. $\hat{\theta} = \theta +$ constant=: $\theta + \theta_o$, so we get from (7)

$$\lambda_s/\mu = p\cos(\theta+\theta_o), \qquad \frac{\lambda^2-\beta}{2} = p\sin(\theta+\theta_o) \tag{8}$$

Now, from the second equation in (8), we get

$$\lambda = \frac{\theta_s}{\mu(\theta)} = \pm\sqrt{2p\sin(\theta+\theta_o)+\beta}\,. \tag{9}$$

so using that $C_s = e^{i\theta}$, we obtain

$$dC = C_s\, ds = \pm\frac{e^{i\theta}\, d\theta}{\mu(\theta)\sqrt{2p\sin(\theta+\theta_o)+\beta}}$$

and so

$$C = C(\theta) = \pm\int^{\theta} \frac{e^{i\tilde{\theta}}}{\mu(\tilde{\theta})\sqrt{2p\sin(\tilde{\theta}+\theta_o)+\beta}}\, d\tilde{\theta}\,. \tag{10}$$

Note also that (9) also gives an expression for the arc-length of C

$$s = \pm\int_{\theta_0}^{\theta_1} \frac{d\theta}{\mu(\theta)\sqrt{2p\sin(\theta+\theta_o)+\beta}}\,.$$

Recall that $\lambda = \kappa/\mu = \theta_s/\mu$ so we obtain from the first equation of (8) that,

$$\frac{1}{\mu}\left(\frac{\theta_s}{\mu}\right)_s = p\cos(\theta+\theta_o).$$

which is an anisotropic version of the *pendulum equation*. In particular, in the isotropic case ($\mu \equiv 1$), we recover, in our setting, *Kirchhoff's kinetic analogy*, which describes the relation between equilibrium equations and the motion of a heavy body turning about a fixed point, i.e. a rigid pendulum [15].

Note that the formula (10) has an interesting linear structure. The only dependency on the choice of γ comes from the appearance of the factor $1/\mu$. However, by (3), this quantity depends linearly on γ. If γ_i, $i = 1, 2$ satisfy the smoothness and convexity conditions that we have imposed then so will $\gamma := a_1\gamma_1 + a_2\gamma_2$ for suitable $a_1, a_2 \approx 0$ and hence, by (10), the elasticae for γ will result from taking the same linear combination of elasticae obtained from the individual γ_i's. Specifically, we take the linear combination of position vectors at points where the oriented tangents of the elasticae agree. In particular, if Ω is replaced by a suitable parallel curve, i.e. γ is replaced by $\gamma + \epsilon$ with ϵ positive and small, then the resulting elasticae will add ϵ times an isotropic elastic curve to an elastica for γ. (See Figure 6).

Below, we relate the arc-length of the curve C with that of Ω.

Proposition 2.2. *Let $\eta(\sigma)$ be an arc-length parameterized arc of the Wulff shape Ω. Let $\beta \in \mathbf{R}$ and $A \in \mathbf{R}^2$ be such that $\beta + A\cdot T > 0$ holds along η. Define a* *reparameterization of* η *by setting*

$$s := \int^{\sigma} \frac{d\sigma}{\sqrt{\beta + A \cdot T}} .$$

Then, there exists an arc-length parameterized (constrained) anisotropic elastic curve $C(s)$ *satisfying*

$$\left(\frac{\lambda_s}{\mu}\right)^2 + \frac{\left(\lambda^2 - \beta\right)^2}{4} = \frac{1}{4}\|A\|^2 .$$

Proof. Notice that since σ is the arc-length parameter of the arc η of Ω, we have that

$$\lambda(s) = \frac{ds}{d\sigma} = \sqrt{\beta + A \cdot T(s)}$$

using the reparameterization of the statement. Moreover, we also have

$$\lambda_s(s) = \kappa \frac{A \cdot N(s)}{2\sqrt{\beta + A \cdot T(s)}} = \frac{1}{2}\kappa \frac{A \cdot N(s)}{\lambda(s)} .$$

Now, a planar curve is an anisotropic elastic curve if and only if it verifies (6). In our setting, using above equations and (9), we conclude that

$$\left(\frac{\lambda_s}{\mu}\right)^2 + \frac{\left(\lambda^2 - \beta\right)^2}{4} = \frac{1}{4}\left(\frac{\kappa}{\mu}\frac{A \cdot N(s)}{\lambda(s)}\right)^2 + \frac{1}{4}\left(A \cdot T(s)\right)^2 = \frac{1}{4}\|A\|^2$$

holds. **q.e.d.**

Remark. The converse to the previous proposition also holds. It will be discussed later in Section 5.1.

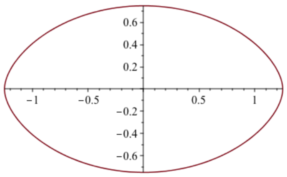
In order to obtain some explicit examples, we consider the densities

$$\gamma_n = \gamma_n(\theta) = 1 + \frac{\cos(n\theta)}{n^2} .$$

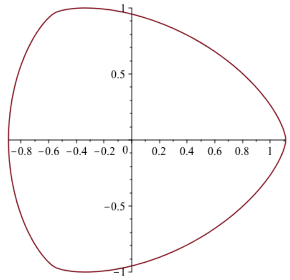
Then

$$\frac{1}{\mu} = \gamma_{\theta\theta} + \gamma = 1 - \left(\frac{n^2 - 1}{n^2}\right)\cos(n\theta) .$$

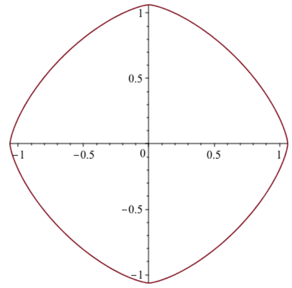
We will refer to the Wulff shape for the density γ_n as Ω_n. Some examples and the corresponding elastic curves, produced using (10), are shown below. Figure 1 shows the Wulff shape for varying values of n. Figure 2 shows orbit like elastica. These have positive curvature everywhere. Figure 3 shows orbit like elastica for γ_4 with varying values of β. Figure 4 shows elasticae with inflection points. Finally, Figure 5 shows anisotropic lemniscates. These are closed elasticae having an inflection point at the double point.



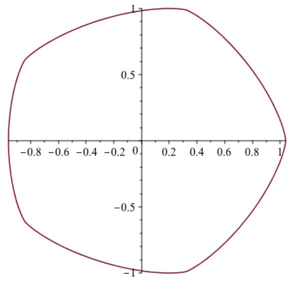
(A) $n = 2$



(B) $n = 3$

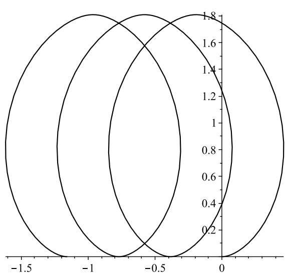


(C) $n = 4$

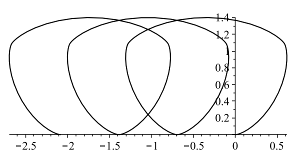


(D) $n = 5$

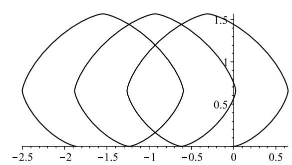
FIGURE 1. The Wulff shapes Ω_n.



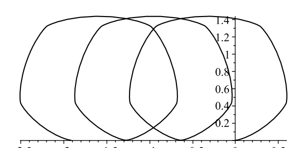
(A) $n = 2$



(B) $n = 3$

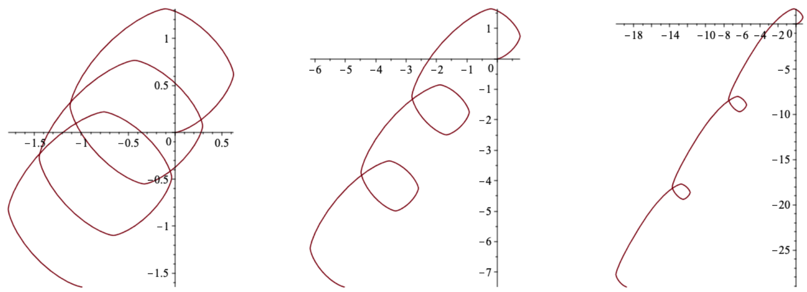


(C) $n = 4$



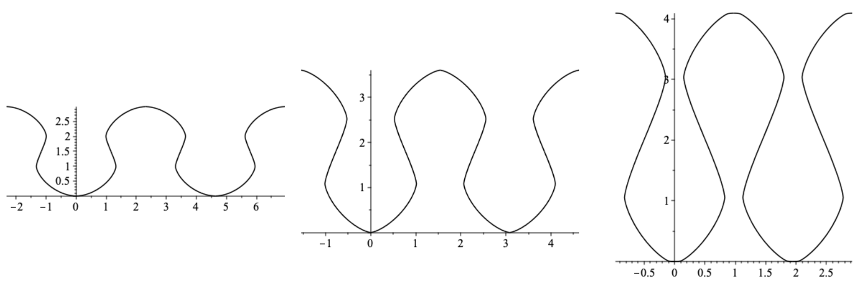
(D) $n = 5$

FIGURE 2. Orbit like anisotropic elasticae for Ω_n.



(A) $\beta = 2$ (B) $\beta = 1.1$ (C) $\beta = 1.001$

FIGURE 3. Orbit like anisotropic elasticae for Ω_4 with $\theta_0 = \pi/3$ and different values of β.



(A) $n = 3$ (B) $n = 4$ (C) $n = 5$

FIGURE 4. Anisotropic elasticae containing inflection points.

3. ANISOTROPIC ENERGY IN 3-SPACE

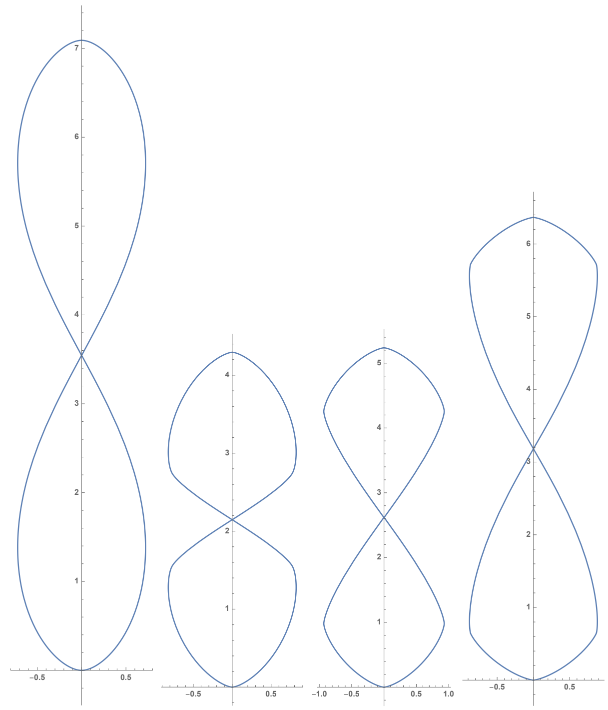
In this section we are going to consider anisotropic energies of a curve $C : I \to \mathbf{R}^3$. The approach will differ substantially from the planar case since the Wulff shape is now two dimensional. Our approach will be to express as much as we can about the equilibrium curves using the Cahn-Hoffman field.

Let $\gamma : S^2 \to \mathbf{R}$ be a smooth, positive function. For each $p \in S^2$, the set of $\vec{v} \in \mathbf{R}^3$ such that $\vec{v} \cdot p \leqslant \gamma(p)$ is a half-space, so the intersection of these half-spaces defines a convex body. The *Wulff shape* W of γ will be defined as the boundary of this convex body:

$$W := \partial \left(\bigcap_{p \in S^2} \{\vec{v} \in \mathbf{R}^3 \mid \vec{v} \cdot p \leqslant \gamma(p)\} \right) .$$

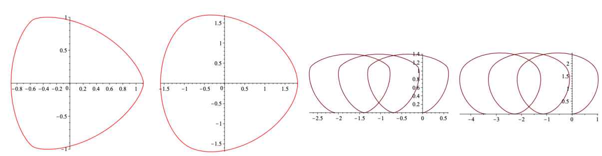
We will assume in this paper that W is smooth. In this case, it is not hard to see that the map

$$\chi : S^2 \to W\,,\ p \mapsto D\gamma(p) + \gamma(p)p$$



(A) $n = 2$ (B) $n = 3$ (C) $n = 4$ (D) $n = 5$

FIGURE 5. Anisotropic lemniscates.



(A) (B) (C) (D)

FIGURE 6. Elastica for a parallel Wulff shape: (A) shows a Wulff shape for a function γ, (B) shows the parallel surface with γ replaced by $\gamma + 0.7$, (C) shows an elastica for the Wulff shape (A) and (D) is an elastica for the Wulff shape (B).

is a smooth bijection, where $D\gamma$ denotes the gradient of γ on S^2. Indeed, the map χ is exactly the inverse of the Gauss map $\nu : W \to S^2$, which is well defined since W is convex.

Since W is convex, at each point $p \in S^2$, we have a bijective, symmetric endomorphism field $d\chi|_p$ given by

$$d\chi|_p = D^2\gamma|_p + \gamma(p)\mathrm{Id} : T_pS^2 \to T_{\chi(p)}W \,, \tag{11}$$

where Id denotes the identity map and $D^2\gamma$ is the Hessian of γ on S^2. Since $T_p(S^2) = T_{\chi(p)}W$, we can also consider $d\chi|_p$ as an endomorphism field $T_{\chi(p)}W \to$

$T_{\chi(p)}W$. Its eigenvalues, denoted by $1/\mu_i$, $i = 1, 2$ are just the principal radii of curvature of W. The orientation of W will always be chosen so that these are positive.

Now, for a smooth curve $C : I \to \mathbf{R}^3$, we define an *anisotropic energy* by

$$\mathcal{F}[C] := \int_C \gamma(T)\, ds\ ,$$

where T denotes the unit tangent to C. We denote the usual Frenet frame along C by $\{T, N, B\}$ (if C is a geodesic, it should be understood that N and B are any unit orthogonal constant vector fields of the normal bundle to C) and consider a variation of C given by

$$(s, \epsilon) \mapsto C(s) + \epsilon\delta C = C(s) + \epsilon[\phi(s)T + \zeta(s)N + \psi(s)B]\ , \tag{12}$$

where s represents the arc-length parameter of C. We then obtain at $(s, \epsilon) = (0, 0)$,

$$\partial_s \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{13}$$

which are precisely the usual Frenet equations involving the curvature, κ, and torsion, τ, of the curve $C(s)$. Moreover, we also obtain the variation of the Frenet frame with respect to δC,

$$\partial_\epsilon \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & [\phi\kappa + \zeta_s - \psi\tau] & [\psi_s + \zeta\tau] \\ -[\phi\kappa + \zeta_s - \psi\tau] & 0 & * \\ -[\psi_s + \zeta\tau] & -* & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}. \tag{14}$$

For the sake of simplicity, we are not explicitly writing the terms $\partial_\epsilon N \cdot B = -\partial_\epsilon B \cdot N$ since they are not used in the paper.

Using the formulas above, we can compute the variation of the measure along the curve,

$$\delta(ds) = (\delta C)_s \cdot T\, ds = [\phi_s - \zeta\kappa]\, ds\ .$$

By using this and integrating by parts, we obtain

$$\begin{aligned} \delta\mathcal{F}[C] &= \int_C D\gamma(T) \cdot \delta T + \gamma[\phi_s - \zeta\kappa]\, ds \\ &= \int_c -\partial_s \big[D\gamma(T) + \gamma(T)T\big] \cdot \delta C\, ds + \big(D\gamma(T) + \gamma(T)T\big|_{\partial C}\ . \end{aligned} \tag{15}$$

If we now define the Cahn-Hoffman field of C by $\xi := \chi \circ T$, we obtain that the Euler-Lagrange equation for the functional $\mathcal{F}$ is $\xi_s = 0$. We therefore define the *anisotropic curvature vector* by the equation

$$\vec{\lambda} := \xi_s\ .$$

Since $\vec{\lambda} \equiv 0$ implies ξ is a constant vector along C, it follows easily that C is a straight line. These are the geodesics in a particularly simple type of Finsler manifold, i.e. one for which the Finsler metric only depends on the direction but not on the position. Such a Finsler manifold is known as a *Minkowski space*.

The Cahn-Hoffman field $\xi = \chi \circ T : I \to W$ can be understood as a curve in the Wulff shape W. The parameter s need not be the arc-length parameterization of the Cahn-Hoffman field, although this may occur in the special case that $\xi(s)$ is a geodesic in W. Indeed, throughout this paper we will denote the arc-length parameter of a curve in W by σ, while reserving s for the arc-length of any curve in $\mathbf{R}^3$. Therefore, for any curve $C(s)$ in $\mathbf{R}^3$ we have an associated curve in the Wulff shape, $\xi(s)$. The converse also holds, although in this case the construction is not independent of the parameterization of the initial curve. For a parameterized curve in W, $\eta(t)$, we define $T(t) := \chi^{-1}(\eta(t)) = \nu(\eta(t))$. Then,

$$C(s) := \int^s T(t)\, dt \tag{16}$$

is an arc-length parameterized curve in $\mathbf{R}^3$ having $T(s)$ as its unit tangent vector field and $\chi(T(s))$ as its Cahn-Hoffman field.

As it turns out, the length of the anisotropic curvature vector of $C(s)$, constructed as above, is related with the choice of parameterization. In our case, since σ represents the arc-length parameter of η, we have

$$\|\vec{\lambda}\|^2 = \|\xi_s\|^2 = \|\xi_\sigma \sigma_s\|^2 = \sigma_s^2 \|\xi_\sigma\|^2 = \sigma_s^2 \,.$$

4. Anisotropic Kirchhoff Elastic Rods

We will now consider a sufficiently smooth arc-length parameterized curve $C(s)$ in $\mathbf{R}^3$. The usual formulation of the variational problem for the Kirchhoff elastic rod (KER), is to add to the elastic energy a second energy coming from an orthonormal framing of the normal bundle of the curve. This energy measures the amount of twisting of the rod. Our treatment here follows closely that of [12], which deals with the isotropic case.

If we denote the normal frame, referred to as the *material frame*, by $\{M_1, M_2\}$ and replace the usual elastic energy by an anisotropic one, we arrive at the functional

$$\mathcal{E}_{\mathcal{K}}[C] := \int_C \left(\|\vec{\lambda}\|^2 + \varpi \|\nabla^\perp M_1\|^2 \right) ds \,, \tag{17}$$

for a suitable real constant ϖ. We will consider this functional as defined on the set of suitably smooth curves having prescribed C^1 boundary conditions at their endpoints.

At this point, we wish to briefly discuss another model for anisotropic rods which appears in the literature (cf. [12] equation (2)). This is defined, relative to an arbitrary framing of the $\perp$ C, by defining $T_s = k_1 M_1 + k_2 M_2$ and using the Lagrangian $a_1 k_1^2 + a_2 k_2^2$ where a_i, $i = 1, 2$ are coupling constants. This is distinct, but related, to our model. If we choose M_i to be a field of eigendirections for $d\chi|_T$, i.e. $d\chi|_T(M_i) = \mu_i^{-1} M_i$, then

$$\|\vec{\lambda}\|^2 = \|d\chi|_T(T_s)\|^2 = \|d\chi|_T(k_1 M_1 + k_2 M_2)\|^2 = \frac{k_1^2}{\mu_1^2} + \frac{k_2^2}{\mu_2^2} \,.$$

The μ_i's are, except in the isotropic case, never constant, but are functions of the tangent direction.

We use the convention that when listing the vectors in any orthonormal frame for $\perp$ C the frame has the same orientation as the Frenet frame $\{N, B\}$. There is another orthonormal frame of $\perp$ C; the parallel frame $\{U_1, U_2\}$. The latter has the property $\nabla^{\perp} U_i \equiv 0$, $i = 1, 2$. Let $\varphi := \angle(N, U_1)$. A simple calculation shows that the parallel frame is determined, up to a constant rotation angle, by the condition

$$\varphi_s = \tau ,$$

where τ is the torsion of the curve C.

Letting $\theta := \angle(M_1, U_1)$, we find that $\|\nabla^{\perp} M_1\|^2 = (\theta_s)^2$. Since the frame can be varied without changing the curve in (17), one easily obtains the condition $\theta_{ss} = 0$ must hold for an equilibrium of the functional, so that $\theta_s \equiv$ constant=: m. We can then write (17) as

$$\mathcal{E}_{\mathcal{K}}[C] := \int_C \left(\|\vec{\lambda}\|^2 + \varpi m^2 \right) \, ds \, . \tag{18}$$

Again following [12], we make the assumption that, as the curve is traversed, the material frame undergoes a fixed number of rotations relative to the Frenet frame. This number q need not necessarily be an integer. This gives

$$2\pi q = \int_C (\varphi - \theta)_s \, ds = \int_C \tau \, ds - mL \, ,$$

where again L denotes the length of C. If we now make a variation of both the curve C and the frame, we arrive at

$$0 = \delta \left(\int_C \tau \, ds \right) + (\delta m) L + m \delta L \, .$$

so

$$\delta m = \frac{1}{L} \delta \left(\int_C \tau \, ds \right) - m \frac{\delta L}{L} \, .$$

If we next take the first variation of the functional in (18) and use the previous equation, we obtain

$$\begin{aligned} \delta \mathcal{E}_{\mathcal{K}}[C] &= \delta \left(\int_C \|\vec{\lambda}\|^2 \, ds \right) + 2\varpi m (\delta m) L + \varpi m^2 \delta L \\ &= \delta \left(\int_C \|\vec{\lambda}\|^2 \, ds + 2\varpi m_0 \int_C \tau \, ds - \varpi m_0^2 \int_C \, ds \right) , \end{aligned}$$

where m_0 denotes the value of m for an extremal configuration. This can be interpreted as meaning that the curve C of an equilibrium configuration, known as a *center line*, is an equilibrium for a functional of the form

$$\mathcal{K}[C] := \int_C \left(\|\vec{\lambda}\|^2 + \alpha\tau + \beta \right) \, ds \, , \tag{19}$$

where α, β are two real constants.

In what follows we will compute the first variation formula of $\mathcal{K}$. By linearity, it is clear that the first variation formula of $\mathcal{K}$ can be computed by parts, i.e.

$$\delta\mathcal{K}[C]=\delta\mathcal{E}_o[C]+\alpha\delta\left(\int_C \tau\, ds\right)+\beta\delta\left(\int_C ds\right),$$

where $\mathcal{E}_o[C]$ denotes the unconstrained *anisotropic elastic energy* in $\mathbf{R}^3$.

We consider,

- **The anisotropic elastic energy:**
  The first variation formula of the anisotropic elastic energy $\delta\mathcal{E}_o[C]$ in terms of an arbitrary parameterization of C can be computed as follows. Using the variation (12),
  $$\mathcal{E}_o[C]=\int_C \frac{\|\xi_t\|^2}{\|C_t\|^2}\,\|C_t\|\, dt=\int_C \frac{\|\xi_t\|^2}{\|C_t\|}\, dt\,.$$
  From this we easily obtain,
  $$\begin{aligned}\delta\mathcal{E}_o[C] &= \int_C 2(\delta\xi)_s\cdot\xi_s-(\delta C)_s\cdot T\|\xi_s\|^2\, ds\\ &= \int_C -2(\delta\xi)\cdot\xi_{ss}+\kappa\zeta\|\xi_s\|^2\, ds+2\delta\xi\cdot\xi_s\big|_{\partial C}\,.\end{aligned}$$
  Since $\xi=\chi(T)$, we get $\delta\xi=d\chi|_T\,(\delta T)$ which can be computed with the aid of (14). Using this, (13) and integrating by parts we obtain
  $$\begin{aligned}\delta\mathcal{E}_o[C] &= \int_C \partial_s\left(2d\chi|_T(\vec{\lambda}_s^T)+\|\vec{\lambda}\|^2T\right)\cdot\delta C\, ds\\ &\quad+\left(2\vec{\lambda}_s\cdot\delta\xi-\left[2d\chi|_T(\vec{\lambda}_s^T)+\|\vec{\lambda}\|^2T\right]\cdot\delta C\right)\big|_{\partial C}\,.\end{aligned}$$
- **The total torsion:**
  For the first variation of the total torsion we first notice that (see formula (26) in the Appendix of [12])
  $$\delta\tau=\left(\frac{1}{\kappa}\,(\delta C)_{ss}\cdot B\right)_s+(\delta C)_s\cdot(\kappa B-\tau T)\,.$$
  Then, by standard arguments involving integration by parts we conclude that
  $$\begin{aligned}\delta\left(\int_C \tau\, ds\right) &= \int_C \delta\tau+\tau\,(\phi_s-\kappa\zeta)\, ds=-\int_C (\kappa B)_s\cdot\delta C\, ds\\ &\quad+\left(\frac{1}{\kappa}\,(\delta C)_{ss}\cdot B+\kappa B\cdot\delta C\right)\Big|_{\partial C}\,.\end{aligned}$$
- **The length functional:**
  Finally, the variation of the length functional can be obtained using formula (15) for $\gamma=1$.

Thus, combining the information above, we end up with the following first variation formula for the anisotropic energy $\mathcal{K}$,

$$\begin{aligned}\delta\mathcal{K}[C] &= \int_C \partial_s \left(2d\chi|_T\left(\vec{\lambda}_s^T\right) + \left[\|\vec{\lambda}\|^2 - \beta\right]T - \alpha\kappa B\right)\cdot\delta C\,ds \\ &\quad \Big(2\vec{\lambda}_s\cdot\delta\xi - (2d\chi|_T(\vec{\lambda}_s^T) + [\|\vec{\lambda}\|^2-\beta]T - \alpha\kappa B)\cdot\delta C \\ &\quad + \frac{\alpha}{\kappa}(\delta C)_{ss}\cdot B\Big)\Big|_{\partial C}.\end{aligned}$$

As a consequence, anisotropic center lines of Kirchhoff elastic rods are described by the conservation law

$$2d\chi|_T\left(\vec{\lambda}_s^T\right) + \left[\|\vec{\lambda}\|^2 - \beta\right]T - \alpha\kappa B \equiv \text{constant} =: A, \tag{20}$$

where $A \in \mathbf{R}^3$ is any constant vector.

Then, since both B and $d\chi|_T\left(\vec{\lambda}_s^T\right)$ are orthogonal to T, by a simple manipulation we get

$$\sigma_s^2 = \|\vec{\lambda}\|^2 = \beta + T\cdot A = \beta + \nu\cdot A, \tag{21}$$

where for the last equality we are considering T as the unit normal to W, ν.

We point out that at any $s_o \in I$ where $\sigma_s^2(s_o) = \|\vec{\lambda}(s_o)\|^2 = 0$ the change of variable from s to σ fails to be bijective and, as a consequence, it does not define a reparameterization. However, these points verify that

$$\vec{\lambda}(s_o) = \xi_s(s_o) = d\chi|_T(T_s(s_o)) = 0\,.$$

Then, since $d\chi|_T$ is an isomorphism we have that $T_s(s_o) = \kappa(s_o)N(s_o) = 0$, concluding that the curvature of C, κ, at these points vanishes, i.e. they represent the *inflection points* of the curve $C(s)$.

Equivalently, looking at the Cahn-Hoffman field in W, ξ, we get that these points are exactly the ones where ξ fails to be regular ($\xi_s(s_o) = 0$). In any case, they are isolated points.

On the other hand, if we consider the normal component of (20), we have

$$2d\chi|_T\left(\vec{\lambda}_s^T\right) - \alpha\kappa B = A - (A\cdot T)\,T.$$

Moreover, if we put $\kappa B = T\times T_s$ and we compose with the differential of the Gauss map of W, $d\nu = d\chi^{-1}$, we can rewrite above equation in terms of operators in W as

$$2\nabla_s^W\xi_s - \alpha d\nu\,(\nu\times\nu_s) = \nabla^W(\nu\cdot A) \tag{22}$$

since now $T = \nu$, the unit normal vector field to W.

Let us now introduce the operator $J = \nu\times\cdot$ (which defines an almost-complex structure on TW) and consider the well known formulas relating J, the Gauss map, ν, the mean curvature of W, H_W, and the Gaussian curvature of W, K_W,

$$\begin{aligned}2H_W J &= -J d\nu - d\nu J \\ 2H_W d\nu &= -d\nu^2 - K_W \mathrm{Id}\,.\end{aligned}$$

The previous formulae follows from the Cayley-Hamilton Theorem. Using these equations and the definition of J, it is easy to see that the second term in (22) becomes

$$\begin{aligned} d\nu\left(\nu \times \nu_s\right) &= d\nu\left(J d\nu(\xi_s)\right) = 2H_W d\nu\left(J(\xi_s)\right) - d\nu^2\left(J(\xi_s)\right) \\ &= K_W\left(J(\xi_s)\right) = K_W \nu \times \xi_s \,. \end{aligned}$$

Thus, substituting above equality in (22), we conclude with the following equation in W

$$2\nabla_s^W \xi_s = \nabla^W\left(\nu \cdot A\right) + \alpha K_W \nu \times \xi_s \,. \tag{23}$$

This equation represents a perturbed second order gradient flow and, therefore, it can be related with a functional in W.

Theorem 4.1. *A smooth curve $\xi : [0, L] \to W$ is the Cahn-Hoffman field of an arc-length anisotropic center line $C(s)$ in $\mathbf{R}^3$ if and only if ξ is a critical point of a functional of the form*

$$\Psi_{A,\alpha}[\xi] = \int_0^L \|\xi_s\|^2 + \nu(\xi(s)) \cdot A \, ds + \alpha \int_{s=0}^{s=L} k_g \, d\sigma \,,$$

where ν is the Gauss map of W, κ_g denotes the geodesic curvature of ξ, σ is the arc-length parameter of ξ and $A \in \mathbf{R}^3$, $\alpha \in \mathbf{R}$ are constants.

Proof. Let $C(s)$ be an arc-length parameterized center line. We wish to express the functional $\mathcal{K}$ using only its Cahn-Hoffman field $\xi(s)$ which requires us to interpret the torsion τ of the curve C in terms of its Cahn-Hoffman field $\xi(s)$. To do this, we use (16) and the well known formula for the torsion of C,

$$\tau = \frac{\det(C_s, C_{ss}, C_{sss})}{\|C_s \times C_{ss}\|^2} \,,$$

to obtain, after some calculation,

$$\tau = \kappa_g \sigma_s + (\arctan(\frac{\tau_g}{\kappa_n}))_s \,. \tag{24}$$

Here, the quantities τ_g and k_n are, respectively, the *geodesic torsion* and *normal curvature* of $\xi(s)$ and σ is the arc-length parameter along ξ. It then follows that

$$\int_0^L \tau \, ds = \int_0^L \kappa_g \, d\sigma + (\arctan(\frac{\tau_g}{\kappa_n})(\sigma(s)))|_0^L \,.$$

Now, taking into account that the endpoints of the center line are fixed to first order and the normal field $\nu(\xi(s))$ is the unit tangent field to C, we must impose the condition

$$\int_0^L \nu(\xi(s)) \, ds \equiv \text{constant.}$$

This involves three scalar constraints which can be included by using a vector Lagrange multiplier $A \in \mathbf{R}^3$. We regard the constant β in (19) as a Lagrange multiplier which fixes the length of C. This corresponds to fixing the parameter domain $[0, L]$ in the functional $\Psi_{A,\alpha}$. We then obtain that if $C(s)$ is a critical point for (19), then $\xi(s)$ is a critical point of a functional $\Psi_{A,\alpha}$. In particular, the Euler-Lagrange equation for $\delta\Psi_{A,\alpha}$ is (23).

Conversely, if $\xi(s)$ is a critical point for a functional $\Psi_{A,\alpha}$, then we can define the arc-length parameterized curve associated to ξ by (16). Then $C_s(s) = T(s) = \nu(\xi(s))$ holds and so $\xi := \chi \circ T$ is the Cahn-Hoffman field of C. For a compactly supported normal variation in W, $\delta\xi = \zeta(s)n$ of ξ, n denoting the normal to ξ in W, we get

$$\delta\left(\int_0^L \kappa_g \, d\sigma\right) = \int_0^L \left(\zeta_{\sigma\sigma} + K_W\zeta\right) \, d\sigma = \int_0^L K_W\zeta \, d\sigma \,, \tag{25}$$

and so taking the first variation of $\Psi_{A,\alpha}$ yields (23). But as we showed above, (23) is equivalent to the conservation law (20) for the curve C. Since this equation characterizes center lines, the result follows. **q.e.d.**

Notice that the first part of above result can also be proven by comparing the Euler-Lagrange equation of $\Psi_{A,\alpha}$ with (23).

The following offers an intrinsic characterization of the Cahn-Hoffman fields of Kirchhoff center lines, i.e. one that only uses the arc-length parameterization.

Theorem 4.2. *Let $A \in \mathbf{R}^3$ and $\beta \in \mathbf{R}$ be constants and let $\eta \subset W$ be a regular curve in W. Assume also that $\beta + A\cdot\nu \geqslant 0$ holds and $\beta + A\cdot\nu \neq 0$ holds almost everywhere on η. Then η is a critical point for the functional $\Psi_{A,\alpha}$ if and only if η is a critical point for the functional*

$$\Phi[\eta] = \int_\eta \sqrt{\beta + \nu\cdot A}\, d\sigma + \frac{\alpha}{2}\int_\eta \kappa_g \, d\sigma \,.$$

In the variational problem for Φ, only variations which preserve the non negativity of $\beta + A\cdot\nu$ near its zeros are allowed.

Proof. First assume that $C(s)$ is an arc-length critical point of $\Psi_{A,\alpha}$. We start by writing equation (23) using the arc-length parameter of the curve ξ in W, σ. Let $n := \nu \times \xi_\sigma$ be the normal to ξ in W. First, notice that $\nabla^W(\nu\cdot A)$ can be decomposed into its tangent and normal components to ξ as,

$$\nabla^W(\nu\cdot A) = (\nu\cdot A)_\sigma \, s_\sigma \xi_s + (\nu\cdot A)_n \, n \,.$$

Then, by (21),

$$\begin{aligned} 2\nabla^W_\sigma \xi_\sigma &= 2s_\sigma^2 \nabla^W_s \xi_s + 2s_{\sigma\sigma}\xi_s = s_\sigma^2\left(\nabla^W(\nu\cdot A) + \alpha K_W \nu\times\xi_s\right) \\ &+ 2s_{\sigma\sigma}\xi_s = s_\sigma^2(\nu\cdot A)_n n + \alpha s_\sigma K_W n \\ &= \left(\frac{(\nu\cdot A)_n}{\beta+\nu\cdot A} + \alpha\frac{K_W}{\sqrt{\beta+\nu\cdot A}}\right) n. \end{aligned} \tag{26}$$

That is, the geodesic curvature of the curve ξ in the Wulff shape W, κ_g, is given by

$$\kappa_g = \partial_n\left(\log(\sqrt{\beta+\nu\cdot A})\right) + \frac{\alpha}{2}\frac{K_W}{\sqrt{\beta+\nu\cdot A}}. \tag{27}$$

Then, for a non negative smooth function f on W, and a smooth compactly supported variation $\delta\xi = \zeta(\sigma)n$, the following formula holds,

$$\delta\left(\int f d\sigma\right) = \int (\zeta\nabla f\cdot n - f k_g\zeta)\, d\sigma = \int \zeta f\left(\nabla \log f\cdot n - k_g\right) d\sigma.$$

At points where $f = 0$, it is understood that $f\nabla\log f$ is replaced by ∇f.

Therefore, if η is a critical point of $\Psi_{A,\alpha}$, we take $f := \sqrt{\beta+\nu\cdot A}$ and using (25), we get

$$\begin{aligned}
\delta\Phi &= \delta\left(\int_\eta \sqrt{\beta+\nu\cdot A}\, d\sigma\right) + \delta\left(\frac{\alpha}{2}\int_\eta \kappa_g\, d\sigma\right)\\
&= \int_\eta \sqrt{\beta+\nu\cdot A}\,\zeta\left(\nabla\log\sqrt{\beta+\nu\cdot A}\cdot n - k_g\right) d\sigma + \frac{\alpha}{2}\int_\eta K_W\zeta d\sigma\\
&= \int_\eta \sqrt{\beta+\nu\cdot A}\,\zeta\left(\nabla\log\sqrt{\beta+\nu\cdot A}\cdot n - k_g + \frac{\alpha}{2}\frac{K_W}{\sqrt{\beta+\nu\cdot A}}\right) d\sigma\\
&= 0
\end{aligned}$$

by (27).

Conversely, if $\delta\Phi = 0$, replacing $\zeta\to\zeta\sqrt{\beta+\nu\cdot A}$, shows that (27) holds and, in turn (26) holds too. On any arc of η where $\beta + A\cdot\nu > 0$, define a new parameter s by

$$ds := \frac{d\sigma}{\sqrt{\beta+\cdot A\cdot\nu}}$$

and consider the associated curve C of ξ as defined in (16). Then, a straightforward calculation using (26) shows that $\xi(s)$ satisfies (23). **q.e.d.**

To finish up this section, we are going to use our previous findings to describe the torsion of the curve $C(s)$ in $\mathbf{R}^3$ in terms of operators in W.

In the proof of Theorem 4.1 we have related the torsion of the curve $C(s)$ in $\mathbf{R}^3$ with the geodesic curvature, κ_g, of the Cahn-Hoffman field in W by the equation $\tau = \kappa_g\sigma_s + \partial_s(\arctan\frac{\tau_g}{\kappa_n})$. Moreover, now, we are going to see how equation (23) can also be used to compute $\tau(s)$.

Using the decomposition of $\nabla^W(\nu\cdot A)$ in W (see previous proof) and (23) we have that

$$\det(\nu,\xi_s,\xi_{ss}) = \det\left(\nu,\xi_s,\nabla^W_s\xi_s\right) = \frac{1}{2}\left((\nu\cdot A)_n\,\sigma_s + \alpha K_W\sigma_s^2\right)\,,$$

where det stands for the *determinant* and where we have used that $\|\nu\times\xi_s\| = \|\nu\|\cdot\|\xi_s\| = \sigma_s$ since ξ_s is tangent to W while ν is its unit normal.

Finally, taking into account the formula for the torsion, (24), and (27) we conclude that

$$\|\vec{\lambda}\|^2\tau = \det(\nu,\xi_s,\xi_{ss}) + \|\vec{\lambda}\|^2\partial_s\left(\arctan\frac{\tau_g}{\kappa_n}\right)\,. \tag{28}$$

5. ANISOTROPIC ELASTIC CURVES

We now specialize our results about anisotropic Kirchhoff elastic rods to study anisotropic elastic curves. If we consider $\alpha = 0$ in $\mathcal{K}$, (19), we obtain the *anisotropic elastic energy* of a curve,

$$\mathcal{E}_\beta[C] := \int_C \left(\|\vec{\lambda}\|^2 + \beta \right) \, ds \, .$$

Specializing (20), we get that the critical points of $\mathcal{E}_\beta$ are characterized by the conservation law

$$2d\chi|_T(\vec{\lambda}_s^T) + \left(\|\vec{\lambda}\|^2 - \beta \right) T \equiv \text{constant} =: A \, . \tag{29}$$

Note that T is orthogonal to the image of $d\chi|_T$ so

$$\|A\|^2 = \|2d\chi|_T(\vec{\lambda}_s^T)\|^2 + \left(\|\vec{\lambda}\|^2 - \beta \right)^2 \equiv \text{constant} =: 4p^2 \, . \tag{30}$$

Clearly $p = 0$ implies $\beta = \|\vec{\lambda}\|^2$ and, in particular, for $\beta = 0$, we get $0 = \vec{\lambda} = \kappa d\chi|_T N$, so C is a line since $d\chi|_T$ is an isomorphism.

In this setting, specializing the result of Theorem 4.1 we have,

Theorem 5.1. *An arc-length parameterized non-linear curve $C(s)$ in $\mathbf{R}^3$ is an anisotropic constrained elastic curve if an only if there is a constant vector $A \in \mathbf{R}^3$ such that the Cahn-Hoffman field $\xi(s)$ is a (non constant) critical point of the functional*

$$\Psi_A[\xi] = \int_0^L \left(\|\xi_s\|^2 + \nu(\xi(s)) \cdot A \right) \, ds$$

for all compactly supported variations of the curve $\xi(s)$ in W. The Euler-Lagrange equation for Ψ is the second order gradient flow

$$2\nabla_s^W \xi_s = \nabla^W (\nu \cdot A) \tag{31}$$

and $\xi(s)$ is a reparametrized geodesic in W if and only if $\nabla^W(\nu \cdot A)$ is tangent to $\xi(s)$.

Proof. First assume that $\xi(s)$ is a a critical point of Ψ_A and so (31) holds. Then there holds

$$\|\xi_s\|^2 \equiv \nu \cdot A + \beta \, , \tag{32}$$

for a constant β. Define the associated arc-length curve of ξ by (16). Then $C_s(s) = T(s) = \nu(\xi(s))$ holds and so $\xi := \chi \circ \nu$ is the Cahn-Hoffman field of C. The equation (31) can be written $2\nabla_s^W \xi_s = d\nu(A - (\nu \cdot A)\nu)$ and, since $d\chi = d\nu^{-1}$, we get

$$2d\chi|_T\left(\nabla_s^W \xi_s\right) + (\nu \cdot A)\nu \equiv A \, .$$

Using (32) and the fact $T = \nu$, we obtain

$$2d\chi|_T\left(\nabla_s^W \xi_s\right) + (\|\xi_s\|^2 - \beta)T \equiv A \, ,$$

so by (29), $C(s)$ is an anisotropic elastic curve.

For the converse, just note that, by reversing the steps above, (29) can be expressed as (31) and so $\xi(s)$ is a critical point of Ψ_A. **q.e.d.**

Recall that $\vec{\lambda} = \xi_s$, so a special case of (29) occurs when the Cahn-Hoffman field is a geodesic parameterized by a constant multiple of arc-length. In this case, we have $\vec{\lambda}_s^T = \nabla_s^W \xi_s = 0$ and $||\vec{\lambda}||^2 - \beta = ||\xi_s||^2 - \beta \equiv 0$.

In this special case, we have the following,

Theorem 5.2. *Let $\eta(\sigma)$ be a minimizing arc-length parameterized geodesic connecting distinct points $\eta(0)$ and $\eta(L)$ in W. Then, the associated curve $C(s)$ in $\mathbf{R}^3$ constructed as in* (16) *satisfies*

$$2\text{dist}(\eta(0), \eta(L)) = \mathcal{E}_1[C] = \int_C \left(||\vec{\lambda}||^2 + 1\right)\, ds \leqslant \mathcal{E}_1[\hat{C}] \tag{33}$$

for all curves $\hat{C}$ in $\mathbf{R}^3$ having the same tangent vectors as C at their endpoints. (The position vectors of the comparison curve $\hat{C}$ do not have to agree with those of C at the endpoints. Here dist *denotes geodesic distance in W.)*

Conversely, if C is a curve in $\mathbf{R}^3$ satisfying the first equality in (33), then C is obtained from the construction given above.

Proof. Let $\iota(\sigma)$, $0 \leqslant \sigma \leqslant \ell$ be any curve in W with $\iota(0) = \eta(0)$ and $\iota(\ell) = \eta(\ell)$. (σ is not necessarily the arc-length parameter of ι). Then

$$2\text{dist}(\eta(0), \eta(L)) \leqslant \int_\iota ||\iota_\sigma||\, d\sigma \leqslant \int_\iota \left(||\iota_\sigma||^2 + 1\right)\, d\sigma\,. \tag{34}$$

Now let $\hat{C}(\sigma)$, $0 \leqslant \sigma \leqslant \ell$ be any arc-length curve in $\mathbf{R}^3$ such that $\hat{C}'(0) = C'(0)$ and $\hat{C}'(\ell) = C'(L)$. Let $\hat{\xi}(\sigma)$ be its Cahn-Hoffman field. Then we can apply (34) with $\iota = \hat{\xi}$ to get

$$2\text{dist}(\eta(0), \eta(L)) \leqslant \mathcal{E}_1[\hat{C}]\,.$$

Note $C_s = T(s)$, in particular, s is the arc-length parameter along C. Also, $\xi(s) = \chi(T(s)) = \eta(s)$, and $\vec{\lambda}(s) = \xi_s(s) = \eta_s(s)$. In particular, $||\vec{\lambda}||^2 \equiv 1$. So for C, we have

$$\mathcal{E}_1[C] = 2\text{dist}(\eta(0), \eta(L)) \leqslant \mathcal{E}_1[\hat{C}]\,.$$

For the converse, observe that $C(s)$ is an arc-length parameterized curve realizing the equality $2\text{dist}(\eta(0), \eta(L)) = \mathcal{E}_1[C]$, then applying (34) to its Cahn-Hoffman field $\xi(s)$, we get that $\xi(s)$ is a minimizing geodesic connecting $\eta(0)$ and $\eta(L)$ and from the second inequality, we get that $||\xi_s||^2 \equiv 1$ holds, so s is also the arc-length parameter of ξ. Since $\xi(s) = \chi(C_s(s))$, the result follows. **q.e.d.**

For values of β, $1 \neq \beta \geqslant 0$, we have the following.

Corollary 5.1. *A curve C is the minimizer of $\mathcal{E}_1$ among all curves having the same tangents as C at their endpoints if and only if for all $r > 0$ the curve $r^{-1}C$ is the minimizer of*

$$\mathcal{E}_{r^2}[\hat{C}] := \int_{\hat{C}} \left(||\vec{\lambda}||^2 + r^2\right)\, ds$$

in the same class of curves.

Furthermore, if p_1 *and* p_2 *are distinct points in* W*, then the minimum value of* $\mathcal{E}_{r^2}$ *among all curves having tangents* $\chi^{-1}(p_i)$*,* $i = 1, 2$ *at their endpoints is* $2r^{-1}\mathrm{dist}(p_1, p_2)$.

Proof. Note that under rescaling

$$\mathcal{E}_1[rC] = \int_C \left(r^{-1} \|\vec{\lambda}\|^2 + r \right) \, ds = r\mathcal{E}_{r^2}[C] \, ,$$

and equivalently

$$\mathcal{E}_1[C] = r\mathcal{E}_{r^2}[r^{-1}C] \, ,$$

From this and the previous theorem, the result follows. **q.e.d.**

Remark. Because of the rescaling, the functionals $\mathcal{E}_r[\cdot]$ for $r < 0$ have no absolute minimum among curves having the same tangents at their endpoints.

An obvious corollary of the previous theorem is the following.

Corollary 5.2. *Let* $\eta(\sigma)$ *be an arc-length parameterized geodesic in the Wulff shape* W*. Then, the associated curve* $C(s)$ *in* $\mathbf{R}^3$ *given by* (16) *is a critical point of*

$$\mathcal{E}_1[C] = \int_C \left(\|\vec{\lambda}\|^2 + 1 \right) \, ds$$

for all variations of C *fixing the endpoints of the curve to first order. (If* C *is closed, the restriction on the variations is to be omitted).*

Proof. Just note that every geodesic in W is locally minimizing. Since the Euler-Lagrange equation for $\mathcal{E}_1$ is local, the result follows. **q.e.d.**

As an illustration of this result let us consider that the Wulff shape W is defined by the following *superellipsoid*

$$W := \{(x, y, z) \in \mathbf{R}^3 \mid \frac{x^q}{a^2} + \frac{y^q}{b^2} + \frac{z^q}{c^2} = 1\} \, ,$$

where a, b, c are positive constants and q is a positive even integer. The reflection (x, y, z) across a coordinate plane is an involutive isometry of $\mathbf{R}^3$ which induces an isometry of W onto itself. It is well known that the the fixed point set of such an isometry is totally geodesic, so the sets of points in W where one of the coordinates vanishes give three geodesics in W. Note that the map $\chi^{-1} : W \to S^2$ is just the Gauss map of W. Since the normal to the level set of a function $f(x, y, z)$ is given, at non critical points of f, by its normalized gradient, we obtain that, the image of each of these geodesics under χ^{-1} is just a great circle in S^2 which lies in one of the coordinate planes. Integrating each of these circles as explained in (16) produces a curve C which is again a circle of radius one in one of the coordinate planes. By the theorem, these circles are critical points for the anisotropic energy functional $\mathcal{E}_1$ corresponding to the superellipsoid Wulff shape W.

We will next use the Cahn-Hoffman map to prove the existence of solutions of the C^1 boundary value problem for the anisotropic bending energy.

Theorem 5.3. *Let* $p_i, i = 1, 2$ *be distinct points in any Wulff shape* W. *Let* ν *denote the Gauss map of* W *and* $\mathcal{S}$ *denote the class of smooth curves* $\xi : [0, L] \to W$ *such that* $\xi(0) = p_1$, $\xi(L) = p_2$

$$\int_0^L \nu(\xi(s))\, ds = Y, \tag{35}$$

where Y *is a fixed vector in* $\mathbf{R}^3$. *Then, if* $\mathcal{S}$ *is not empty, there exists an arc-length anisotropic elastic curve* $C(s)$ *in* $\mathbf{R}^3$ *with endpoints* 0 *and* Y *and having tangent vector* $\nu(p_1)$ *at* 0 *and* $\nu(p_2)$ *at* Y.

Proof. We consider the functional assigning to any curve $\xi(s) \in \mathcal{S}$ its energy

$$\Psi[\xi] = \int_0^L ||\xi_s(s)||^2\, ds\,.$$

Choose a sequence $\{\xi_n(s)\}$ such that $\Psi[\xi_n]$ converges to the infimum of the energy over $\mathcal{S}$. By the Arzela-Ascoli Theorem, it can be assumed, by passing to a subsequence if necessary, that ξ_n converges uniformly on $[0, L]$ to a continuous curve $\xi^*(s)$. Since $\nu : W \to S^2$ is smooth, it is clear that ξ^* satisfies the condition (35).

Also, by applying the Banach-Alaoglu Theorem, we can also assume that ξ_n converges weakly in $H^1[0, L]$ to ξ^*, where $H^1[0, L]$ denotes the first Sobolev space. We now want to assert that for a fixed vector $\Lambda \in \mathbf{R}^3$, which serves as a Lagrange multiplier, ξ^* is a weak solution of the variational problem (see Corollary 5.1)

$$\delta \left(\int_0^L ||\xi_s(s)||^2 + \Lambda \cdot \nu(\xi(s))\, ds \right) = 0\,.$$

In order to assert this, it is necessary to show that $\mathcal{S}$ has the right "manifold" property that for any $\xi(s) \in \mathcal{S}$ and any "tangent vector" $\dot{\xi}$ which is a smooth section of the bundle $\xi^*(TW)$, vanishing at 0 and L and satisfying the linearized condition

$$\delta \left(\int_0^L \nu(\xi(s))\, ds \right) = \int_0^L d\nu_{\xi(s)}(\dot{\xi}(s))\, ds = 0\,, \tag{36}$$

there exists a smooth curve $\xi(s, \epsilon) = \xi(s) + \epsilon\dot{\xi}(s) + \mathcal{O}(\epsilon^2)$ in $\mathcal{S}$.

Instead of working with the curve $\xi(s)$ in W, we will work with its image $\nu(s) := \nu(\xi(s))$ in S^2. Set $\dot{\nu}(s) := d\nu_{\xi(s)}(\dot{\xi}(s))$ which is a section of $\nu^*(TS^2)$. The condition (36) is then just

$$\int_0^L \dot{\nu}\, ds = 0\,.$$

We claim that there exist fields $\eta_i(s) \in \nu^*(TS^2)$ which vanish at 0 and L such that the matrix

$$\left(\int_0^L \eta_1(s)\, ds \quad \int_0^L \eta_2(s)\, ds \quad \int_0^L \eta_2(s)\, ds \right)$$

is invertible. Let $E_1^T = E_1 - \nu_1 \nu$ denote the tangential part of E_1. Then

$$\int_0^L E_1^T \, ds = \begin{pmatrix} L - \int_0^L \nu_1^2 \, ds \\ -\int_0^L \nu_1 \nu_2 \, ds \\ -\int_0^L \nu_1 \nu_3 \, ds \end{pmatrix} \neq \vec{0}$$

since the first component is non zero. Multiplying by a suitable function $\zeta(s)$ which vanishes at 0 and L, we define $\eta_1 := \zeta$

$$\vec{V}_1 := \int_0^L \eta_1(s) \, ds \neq \vec{0} \, .$$

Note that $\nu \times \vec{V}_1$ is a section of $\nu^*(TS^2)$ and lies in the plane perpendicular to $\vec{V}_1$. For a suitable function $\zeta_1(s)$, we have

$$\int_0^L \zeta_1 \nu \times \eta_1 \, ds \neq \vec{0}$$

and we define $\eta_2 = \zeta_1 \nu \times \eta_1$ and

$$\vec{V}_2 := \int_0^L \eta_2(s) \, ds \, .$$

Note $\vec{V}_2$ is orthogonal to $\vec{V}_1$. Let $\vec{V} := \vec{V}_1 \times \vec{V}_2$. On any arc ι of $\nu(s)$, there is a section of η of $\nu^*(TS^2)$ with $0 \neq \eta \cdot \vec{V} \geqslant 0$. Let ζ_2 be a non negative function with compact support in ι. Then defining $\eta_3 := \zeta_2 \eta$ shows that the claim holds.

We now apply an argument due to Bolza. Form the variation

$$\tilde{\nu}(\epsilon, \epsilon_1, \epsilon_2, \epsilon_3) := \exp^{S^2}_{\nu(s)} \left(\epsilon \dot{\nu} + \sum_{i=1}^{3} \epsilon_i \eta_i \right)$$

and consider the map $F : \mathbf{R}^4 \to \mathbf{R}$ defined by

$$(\epsilon, \epsilon_1, \epsilon_2, \epsilon_3) \mapsto \int_0^L \tilde{\nu} \, ds \, .$$

Then ,

$$\left(\partial_{\epsilon_1} F, \partial_{\epsilon_2} F, \partial_{\epsilon_3} F\right)\big|_{(0,0,0,0)} = \left(\int_0^L \eta_1(s) \, ds \quad \int_0^L \eta_2(s) \, ds \quad \int_0^L \eta_2(s) \, ds \right) ,$$

which is non singular. By the Implicit Function Theorem, it is possible to solve $\eta_i = \eta_i(\epsilon)$, $i = 1, 2, 3$ such that

$$\int_0^L \tilde{\nu}(\epsilon, \epsilon_1(\epsilon), \epsilon_2(\epsilon), \epsilon_3(\epsilon)) \, ds \equiv Y \, .$$

We can now assert that ξ^* is a weak solution of an equation

$$2\nabla_s^W \xi_s^* = \nabla^W \nu(\xi^*(s)) \cdot \Lambda \, ,$$

i.e.

$$\int_0^L \left(2\xi_s^* \cdot \dot{\xi}_s + \dot{\xi} \cdot \nabla^W \nu(\xi^*(s)) \cdot \Lambda \right) \, ds = 0$$

for variations $\dot{\xi}(s)$ vanishing at 0 and L. Note that $\nabla^W \nu(\xi^*(s)) \cdot \Lambda$ is a continuous vector field on W from which it follows that ξ^* is equivalent to a classical C^2 solution.

Define the associated arc-length curve of ξ by (16). Then by Theorem 5.1 , $C(s)$ is an anisotropic elastic curve. **q.e.d.**

Remark. There is, in general, no uniqueness of the solution. Even in the planar, isotropic case, there are examples of C^1 initial data for which up to four distinct solutions are known.

5.1. **Planar Curves.** The results above may be specialized to study curves in $\mathbf{R}^2$. Then, we should recover the results of Section 2. Any one dimensional Wulff shape, denoted throughout the paper by Ω, can always be realized as the intersection of a bidimensional Wulff shape, W, with a suitable plane. Precisely, $\Omega = W \cap \pi$ where π is the affine plane passing through the point $\nu(s)$ and with directions $T(s)$ and $d\chi|_T(N)$, where N denotes the normal to the curve. Then, we can compute the anisotropic curvature vector, obtaining

$$\vec{\lambda} = \xi_s = d\chi|_T(T_s) = \kappa d\chi|_T(N).$$

Recall that from (11), at each $T \in S^2$ we have

$$\vec{\lambda} = \kappa \left(D^2\gamma|_T + \gamma(T)\mathrm{Id}\right) N = \kappa \left([\gamma|_T]_{\theta\theta} + \gamma|_T\right) N\,,$$

where θ is the angle that the tangent forms with the positive part of the horizontal axis.

Now, since $C(s)$ is a planar curve, $T \in S^1 \subset S^2$ and, therefore, $\gamma|_T \colon S^1 \to \mathbf{R}^+$. Thus, defining the function μ by $1/\mu := [\gamma|_T]_{\theta\theta} + \gamma|_T$ and using (9) we conclude that

$$\vec{\lambda} = \frac{\kappa}{\mu} N = \lambda N\,,$$

where λ is the (scalar) anisotropic curvature of the plane curve $C(s)$.

Finally, from above equation it is clear that $\vec{\lambda}_s^{\perp} = \lambda_s N$. Then, substituting in (30) we get

$$\|A\|^2 = 4\|d\chi|_T(\lambda_s N)\|^2 + \left(\lambda^2 - \beta\right)^2 = 4\left(\frac{\lambda_s}{\mu}\right)^2 + \left(\lambda^2 - \beta\right)^2 = 4p^2\,.$$

That is, we recover equation (6) of Section 2, as desired.

6. Rotationally Symmetric Wulff Shape

Throughout this section, we consider a rotationally symmetric Wulff shape, $W \subset \mathbf{R}^3$. In this case, the eigendirections of $d\chi|_T$ are given by the vectors E_3^T with eigenvalue $1/\mu_1$ and $T \times E_3^T$, $\times$ denoting the cross product, with eigenvalue $1/\mu_2$. Both eigenvalues depend only on $T_3 = \nu_3$.

Moreover, these eigenvalues can be computed from the density function that defines W, γ, obtaining

$$\frac{1}{\mu_1} = \left(1 - \nu_3^2\right)\gamma''(\nu_3) + \frac{1}{\mu_2}\,, \qquad \frac{1}{\mu_2} = \gamma(\nu_3) - \nu_3\gamma'(\nu_3)\,. \tag{37}$$

Locally, we are going to choose spherical coordinates in $\mathbf{R}^3$, so that the unit normal to W can be written as

$$\nu = (\sin\varphi\cos\vartheta, \sin\varphi\sin\vartheta, \cos\varphi)\ ,$$

for $\vartheta \in (0, 2\pi)$ and $\varphi \in (0, \pi)$. Notice that, $\nu_3 = \cos\varphi$, so in these coordinates the principal curvatures of W only depend on the parameter φ.

With respect to the local parameterization of W defined by above expression of ν, one can obtain all the geometric information of the Wulff shape W. In particular, we have that the *Gauss-Codazzi equations* of W reduce to just

$$\frac{\cos\varphi}{\mu_1} = \left(\frac{\sin\varphi}{\mu_2}\right)_\varphi\ , \tag{38}$$

which allows us to compute the derivative of μ_2 in terms of well known quantities on W.

Now, let $\xi : I \to W \subset \mathbf{R}^3$ be a smooth curve in W locally parameterized as $\xi(s) = (\varphi(s), \vartheta(s))$. Then, we have the sufficient and necessary conditions for $\xi(s)$ to be the Cahn-Hoffman field of an anisotropic center line.

Proposition 6.1. *The curve $\xi(s) = (\varphi(s), \vartheta(s))$ in W is the Cahn-Hoffman field of an arc-length anisotropic center line in $\mathbf{R}^3$ for a constant vector A in* (20)*, if and only if, the following equations are verified*

$$\left(\frac{\varphi_s}{\mu_1}\right)_s - \frac{\vartheta_s^2}{\mu_2}\sin\varphi\cos\varphi - \frac{\mu_1}{2}\left(\nu_\varphi\cdot A - \alpha\vartheta_s\sin\varphi\right) = 0$$
$$\left(\frac{\vartheta_s}{\mu_2}\right)_s \sin^2\varphi + 2\frac{H_W}{K_W}\vartheta_s\varphi_s\sin\varphi\cos\varphi - \frac{\mu_2}{2}\left(\nu_\vartheta\cdot A + \alpha\varphi_s\sin\varphi\right) = 0,$$

where K_W and H_W are the Gaussian and mean curvatures of W, respectively.

Proof. Let $\xi(s) = (\varphi(s), \vartheta(s))$ be a parameterized regular curve in a rotationally symmetric Wulff shape, W. Then, by the linearity of $d\chi|_T$, we have that

$$\xi_s = d\chi|_T(\nu_s) = d\chi|_T(\nu_\varphi\varphi_s + \nu_\vartheta\vartheta_s) = \frac{\varphi_s}{\mu_1}\nu_\varphi + \frac{\vartheta_s}{\mu_2}\nu_\vartheta\ ,$$

since ν_φ and ν_ϑ are the corresponding eigendirections of $d\chi|_T$.

Now, recall that from Theorem 4.1, $\xi(s)$ is the Cahn-Hoffman field of an arc-length parameterized anisotropic center line $C(s)$ in $\mathbf{R}^3$ if and only if ξ is a critical point of $\Psi_{A,\alpha}$. That is, if and only if ξ verifies its corresponding Euler-Lagrange equation, (23).

Therefore, in terms of the local parameterization $\xi(s) = (\varphi(s), \vartheta(s))$ we have

$$2\nabla_s^W\xi_s = 2\left(\frac{\varphi_s}{\mu_1}\right)_s\nu_\varphi + 2\left(\frac{\vartheta_s}{\mu_2}\right)_s\nu_\vartheta + 2\frac{\varphi_s}{\mu_1}\nabla_s^W\nu_\varphi + 2\frac{\vartheta_s}{\mu_2}\nabla_s^W\nu_\vartheta\ .$$

Then, since $(\nu_\varphi)_s = -\varphi_s \nu + \vartheta_s \frac{\cos\varphi}{\sin\varphi}\nu_\vartheta$ and $(\nu_\vartheta)_s = -\vartheta_s \sin^2\varphi \nu_\varphi + \varphi_s \frac{\cos\varphi}{\sin\varphi}\nu_\vartheta$, considering the tangent components to W, we conclude that

$$\begin{aligned} 2\nabla_s^W \xi_s &= \left[2\left(\frac{\varphi_s}{\mu_1}\right)_s - 2\frac{\vartheta_s^2}{\mu_2}\sin\varphi\cos\varphi\right]\nu_\varphi \\ &+ \left[2\left(\frac{\vartheta_s}{\mu_2}\right)_s + 4\frac{H_W}{K_W}\vartheta_s\varphi_s\frac{\cos\varphi}{\sin\varphi}\right]\nu_\vartheta\,. \end{aligned}$$

On the other hand, we get that

$$\nabla^W(\nu\cdot A) = \nu_\varphi \cdot A\xi_\varphi + \nu_\vartheta\cdot\xi_\vartheta = \mu_1\nu_\varphi\cdot A\nu_\varphi + \mu_2\nu_\vartheta\cdot\nu_\vartheta\,.$$

Finally, it is a straightforward computation to check that $\sin\varphi\nu\times\nu_\varphi = \nu_\vartheta$ and $\nu\times\nu_\vartheta = -\sin\varphi\nu_\varphi$, where again $\times$ denotes the cross product in $\mathbf{R}^3$. Therefore, using these relations we obtain,

$$\nu\times\xi_s = \frac{\varphi_s}{\mu_1}\nu\times\nu_\varphi + \frac{\vartheta_s}{\mu_2}\nu\times\nu_\vartheta = \frac{\varphi_s}{\mu_1\sin\varphi}\nu_\vartheta - \frac{\vartheta_s\sin\varphi}{\mu_2}\nu_\varphi\,.$$

That is, after combining everything, we see that equation (23) is equivalent to the system of equations of the statement, since they are precisely the ν_φ and ν_ϑ components, up to a constant. This finishes the proof. **q.e.d.**

In particular, above result is quite illuminating when $A = rE_3$, for some real constant $r\in\mathbf{R}$.

Corollary 6.1. *Assume that* $A = rE_3$ *in* (20) *for some* $r\in\mathbf{R}$. *Then,* $\xi(s) = (\varphi(s),\vartheta(s))$ *is the Cahn-Hoffman field of an arc-length anisotropic center line in* $\mathbf{R}^3$ *for the constant vector* A, *if and only if, the following equations are verified*

$$\begin{aligned} &\left(\frac{\varphi_s}{\mu_1}\right)_s - \frac{\vartheta_s^2}{\mu_2}\sin\varphi\cos\varphi + \frac{\mu_1}{2}(r+\alpha\vartheta_s)\sin\varphi = 0 \\ &\left(\frac{\vartheta_s}{\mu_2}\right)_s \sin^2\varphi + \left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)\vartheta_s\varphi_s\sin\varphi\cos\varphi - \frac{\alpha}{2}\mu_2\varphi_s\sin\varphi = 0. \end{aligned}$$

Proof. In this case, since $A = rE_3$, we have that

$$\nabla^W(\nu\cdot A) = r\nabla^W(\nu_3) = r\nabla^W(\cos\varphi) = -r\sin\varphi\xi_\varphi = -r\mu_1\sin\varphi\nu_\varphi\,.$$

That is, $\nu_\varphi\cdot A = -r\sin\varphi$ and $\nu_\vartheta\cdot A = 0$. Then, substituting it in the system of equations of previous proposition we finish the proof. **q.e.d.**

Under the assumptions of this last result we are in conditions to obtain another constant of motion for center lines.

Proposition 6.2. *Consider that the constant of motion* (20) *verifies that* $A = rE_3$ *for some* $r\in\mathbf{R}$. *Then, an arc-length parameterized center line* $C(s)$ *in* $\mathbf{R}^3$ *also admits the following constant of motion*

$$\frac{\det(\nu,\xi_s,\xi_{ss})}{K_W} - \frac{\alpha}{4}\|\vec{\lambda}\|^2 \equiv \text{constant.}$$

Proof. Let us begin by recalling that, in general, the torsion of an arc-length parameterized center line in $\mathbf{R}^3$ is related with its Cahn-Hoffman field and operators in W, as proven in (28). Therefore, using the local parameterization of ξ given by $\xi(s) = (\varphi(s), \vartheta(s))$ and computations of previous proof we have that

$$\begin{aligned}
\det(\nu, \xi_s, \xi_{ss}) &= (\nu \times \xi_s) \cdot \xi_{ss} = (\nu \times \xi_s) \cdot \nabla_s^W \xi_s \\
&= \frac{\mu_1 (r + \alpha \vartheta_s)}{2\mu_2} \vartheta_s \sin^2 \varphi + \alpha \frac{\mu_2}{2\mu_1} \varphi_s^2 \\
&= K_W \left(r \frac{\vartheta_s \sin^2 \varphi}{2\mu_2^2} + \frac{\alpha}{2} \|\xi_s\|^2 \right) .
\end{aligned}$$

As a consequence, using that $\vec{\lambda} = \xi_s$,

$$\frac{\det(\nu, \xi_s, \xi_{ss})}{K_W} - \frac{\alpha}{4} \|\vec{\lambda}\|^2 = r \frac{\vartheta_s \sin^2 \varphi}{2\mu_2^2} + \frac{\alpha}{4} \|\xi_s\|^2 .$$

By linearity, we can differentiate each term independently with respect to s. On one hand,

$$\left(\|\xi_s\|^2 \right)_s = 2\xi_s \cdot \xi_{ss} = 2\xi_s \cdot \nabla_s^W \xi_s = -r \varphi_s \sin \varphi ,$$

where both equations of Corollary 6.1 have been used to simplify the result.

On the other hand, taking into account the Codazzi equation of the Wulff shape W, (38), we prove that

$$\left(\frac{1}{\mu_2} \right)_s = \left(\frac{1}{\mu_2} \right)_\varphi \varphi_s = \left(\frac{1}{\mu_1} - \frac{1}{\mu_2} \right) \varphi_s \frac{\cos \varphi}{\sin \varphi} ,$$

and, therefore

$$\begin{aligned}
\left(\frac{\vartheta_s \sin^2 \varphi}{\mu_2^2} \right)_s &= \left(\frac{\vartheta_s}{\mu_2} \right)_s \frac{\sin^2 \varphi}{\mu_2} + 2 \frac{\varphi_s \vartheta_s}{\mu_2^2} \sin \varphi \cos \varphi + \frac{\vartheta_s \sin^2 \varphi}{\mu_2} \left(\frac{1}{\mu_2} \right)_s \\
&= \frac{1}{\mu_2} \left[\left(\frac{\vartheta_s}{\mu_2} \right)_s \sin^2 \varphi + \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \varphi_s \vartheta_s \sin \varphi \cos \varphi \right] \\
&= \frac{\alpha}{2} \varphi_s \sin \varphi ,
\end{aligned}$$

where in the last equality we have used the second equation of Corollary 6.1.

Finally, combining both equalities we get that the derivative is zero, so the constant of motion is verified. **q.e.d.**

Notice that with the aid of the constants of motion, the system of equations of Corollary 6.1 can be integrated once. Indeed, in next proposition we describe the first integrals.

Proposition 6.3. *The curve $\xi(s) = (\varphi(s), \vartheta(s))$ is the Cahn-Hoffman field of an arc-length anisotropic center line in $\mathbf{R}^3$ for the constant vector $A = rE_3$, if and* *only if, the following equations are verified*

$$\frac{\varphi_s^2}{\mu_1^2}+\frac{\vartheta_s^2}{\mu_2^2}\sin^2\varphi-r\cos\varphi=\text{constant}=:\beta \tag{39}$$

$$\frac{\vartheta_s\sin^2\varphi}{\mu_2^2}+\frac{\alpha}{2}\cos\varphi\equiv\text{constant}=:b\,. \tag{40}$$

Proof. For any anisotropic center line in $\mathbf{R}^3$ we have that the following quantity must be constant (see equation (21))

$$\|\vec{\lambda}\|^2-\nu\cdot A=\beta\,,$$

where $\beta\in\mathbf{R}$ denotes the constant appearing in (19).

Thus, in our rotationally symmetric setting, for $A=rE_3$ we get

$$\frac{\varphi_s^2}{\mu_1^2}+\frac{\vartheta_s^2}{\mu_2^2}\sin^2\varphi-r\cos\varphi=\beta\,,$$

as desired.

Now, for the second equation we consider the result of previous proposition

$$\frac{\det\left(\nu,\xi_s,\xi_{ss}\right)}{K_W}-\frac{\alpha}{4}\|\vec{\lambda}\|^2=r\frac{\vartheta_s\sin^2\varphi}{2\mu_2^2}+\frac{\alpha}{4}\|\vec{\lambda}\|^2\equiv\text{constant}\,.$$

By a simple manipulation, one can see that this yields to (40), proving the result.

In fact, as a remark, notice that differentiating both equations, (39) and (40), we obtain the system of Corollary 6.1. This is a straightforward way of proving the statement. **q.e.d.**

We will now use (39)-(40) to give quadrature formulas for constructing certain anisotropic elasticae. We represent the Wulff shape W in the form

$$\chi=(ue^{\imath\vartheta},V)\,,$$

where we have identified the horizontal plane with the complex plane $\mathbf{C}$. We work in the part of W where $V=V(u)$ holds. Then, when $\alpha=0$ holds, the equations (39)-(40) are equivalent to

$$\left(1+V_u^2\right)u_s^2+u^2\vartheta_s^2-\frac{r}{\sqrt{1+V_u^2}}\equiv\beta\,,\qquad \vartheta_s u^2\equiv b\,. \tag{41}$$

Using the second equation to solve for s_ϑ, we get after some manipulation

$$\left(1+V_u^2\right)u_\vartheta^2+u^2=\left(\beta+\frac{r}{\sqrt{1+V_u^2}}\right)s_\vartheta^2=\left(\beta+\frac{r}{\sqrt{1+V_u^2}}\right)\frac{u^4}{b^2}\,.$$

This leads to a quadrature formula for ϑ.

$$\pm\,d\vartheta=\frac{\sqrt{1+V_u^2}\,du}{u\sqrt{\left[\beta+\frac{r}{\sqrt{1+V_u^2}}\right]\frac{u^2}{b^2}-1}} \tag{42}$$

Once ϑ has been found, we obtain an integral formula for the curve C

$$C = \int dC = \int \nu \, ds = \int \frac{\left(-V_u e^{i\vartheta}, 1\right)}{\sqrt{1+V_u^2}} s_\vartheta \vartheta_u \, du$$
$$= \int \frac{\left(-V_u e^{i\vartheta}, 1\right)}{\sqrt{\left[\beta + \frac{r}{\sqrt{1+V_u^2}}\right] \frac{u^2}{b^2} - 1}} \frac{u}{b} \, du \, ,$$

Finally, in order to obtain some explicit examples, we take a Wulff shape which is the surface W obtained by taking $\gamma := \nu_3^{-1} - \nu_3$, for $\nu_3 > 0$. This results in the paraboloid given by

$$x_3 = -\frac{x_1^2 + x_2^2}{4} \, .$$

Although this surface is not closed, any compact piece of an anisotropic centerline (or, in particular, an anisotropic elastica) for this Wulff shape lies in a compact domain of W which can be realized as a domain in a closed Wulff shape obtained by attaching a cap to the compact domain, denoted again by W. This choice of a Wulff shape lends itself to simplified calculation.

In this case, it is easy to check, using (37), that the principal curvatures are

$$\mu_1(\varphi) = \cos^3 \varphi/2 \, , \qquad \mu_2(\varphi) = \cos \varphi/2 \, .$$

Then, solving the system of equations (39)-(40) and taking into account the construction of the associated curve in $\mathbf{R}^3$, (16), we can get some examples. For instance, in Figure 7 we illustrate some anisotropic elastic curves where the associated Cahn-Hoffman field is a geodesic in the Wulff shape, i.e. for the constant of motion $A = rE_3 = 0$. Moreover, in Figure 8 we plot an anisotropic elastica for $A = rE_3$ with $r = 2$.

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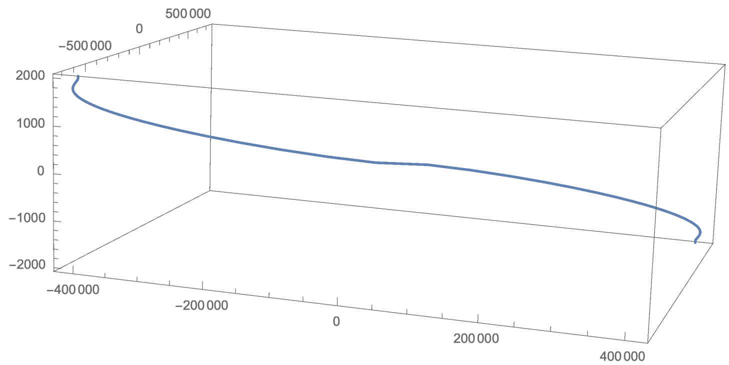
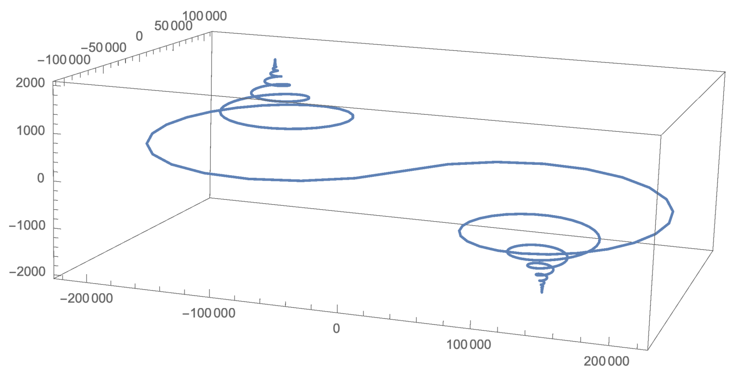


(A) $b = 3$



(B) $b = 30$

FIGURE 7. Anisotropic elastic curves for the paraboloid as Wulff shape. Here, $\beta = 1$ and $\alpha = r = 0$.

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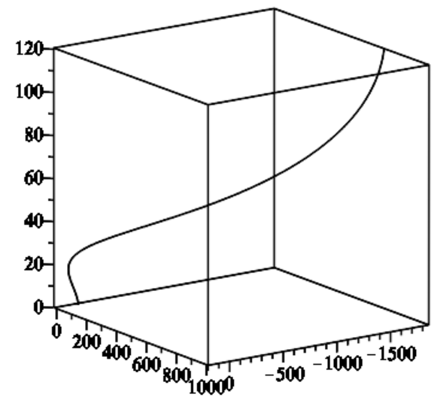

FIGURE 8. Anisotropic elastica for the paraboloid W with $\beta = 1$, $r = 2$ and $\alpha = 0$.

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