Higher Order Variations of Constant Mean Curvature Surfaces

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Abstract

We study the third and fourth variation of area for a compact domain in a constant mean curvature surface when there is a Killing field on $\mathbf{R}^3$ whose normal component vanishes on the boundary. Examples are given to show that, in the presence of a zero eigenvalue, the non negativity of the second variation has no implications for the local area minimization of the surface.

1 Introduction

Many mathematical and physical problems involve the determination of the absolute local or global minimizer for a given variational problem. Once an equilibrium (critical point) has been found, the second variation (hessian) is a proven tool to test if minimization has been achieved. In some cases however, the second variation formula may not be sufficient to determine if an equilibrium is even a weak local minimum of a variational problem. This is, of course, already evident in finite dimensional problems in the case where the second derivative is degenerate. It is this issue which we address here for the constrained variational problem in the three-dimensional euclidean space $\mathbf{R}^3$ which gives rise to constant mean curvature (CMC) surfaces.

We will consider a compact CMC surface represented by an immersion $X : \Sigma \to \mathbf{R}^3$, where Σ is an oriented smooth two dimensional manifold with smooth boundary $\partial\Sigma$. We will assume that X is *critical.* By this we mean that the second eigenvalue of the Jacobi operator is zero. In addition, we assume that the Jacobi field(s) for which this minimum achieved, is induced by a one parameter group of isometries of $\mathbf{R}^3$. We call such a field a *Killing-Jacobi field* and the critical immersion will then be called *Killing critical*. Our analysis then consists of deriving third and fourth order variational formulas to determine if minimization has taken place.

We apply these formulas to study Killing critical domains in unduloids and nodoids. CMC surfaces of revolution is called Delaunay surfaces. Among them, unduloids are periodic surfaces without self intersections, and nodoids are periodic surfaces with self intersections. In particular, we produce examples which show that when the second variation of area is non negative for all volume preserving variations of the surface, the surface may locally minimize area but need not do so.

This paper is organized as follows. Section 2 contains preliminaries about the second variation of area for CMC surfaces. In particular, we give definitions of weak stability, strict stability and instability for CMC surfaces (Definition 2.1), and also we give criteria of stability (Theorem 2.1). Section 3 contains calculations of the third and fourth variation of area. Section 4 contains a discussion of conjugate Delaunay surfaces. We give explicit representations of an unduloid and its conjugate nodoid (Proposition 4.1), which is essential to the calculations of examples in the following two sections. We remark that a CMC surface and its conjugate are locally isometric. Sections 5 and 6 contain analyses of the stability and the fourth variation of area for specific domains in unduloids and nodoids, respectively. In Section 5, we prove that one period of an unduloid $\mathcal{U}$ bounded by two consecutive necks is (weakly) stable (Theorem 5.1). However, for any nontrivial volume-preserving variations for which the second variation of area vanishes, the third variation of area is zero (Proposition 5.2) and the fourth variation is negative (Theorem 5.2). This means that $\mathcal{U}$ is not a local minimizer of area for a volume-preserving variation fixing the boundary. In Section 6, we study the part of a nodoid $\mathcal{N}$ which lies between two circles of points having horizontal tangent planes, here "horizontal" means that the plane is orthogonal to the axis of revolution of $\mathcal{N}$. We prove that $\mathcal{N}$ is (weakly) stable (Proposition 6.1). And, for any nontrivial volume-preserving variations for which the second variation of area vanishes, the third variation of area is zero (Lemma 6.2) and the fourth variation is positive (Theorem 6.1). This means that $\mathcal{N} = X(\Sigma)$ is a minimizer of area for any volume-preserving variation $\{X(\epsilon)\}_\epsilon \subset C^{2+\alpha}(\Sigma, \mathbf{R}^3)$ of X such that $X(\epsilon)|_{\partial\Sigma} = X|_{\partial\Sigma}$ holds.

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2 Preliminaries

Let Σ denote an oriented smooth compact surface with smooth boundary $\partial\Sigma$ and let $X : \Sigma \to \mathbf{R}^3$ be a smooth (up to $\partial\Sigma$) immersion with constant mean curvature (CMC) $H \neq 0$ and normal map $\nu : \Sigma \to S^2 := \{x \in \mathbf{R}^3 \mid \|x\| = 1\}$. It is well known that CMC surfaces arise variationally as follows. A smooth variation $X(\epsilon)$ of X is said to be admissible if $X(\epsilon)|_{\partial\Sigma} = X|_{\partial\Sigma}$ is satisfied for all ϵ. Consider an admissible variation $X(\epsilon)$ of X. We denote differentiation with respect to ϵ using "prime". For a general immersion $X : \Sigma \to \mathbf{R}^3$, $X(\epsilon)$ is given by

$$X(\epsilon) = X + \epsilon X'(0) + \mathcal{O}(\epsilon^2) \tag{1}$$

with $X'(0)|_{\partial\Sigma} \equiv 0$. The first variation of area is given by

$$\mathcal{A}'(0) = -\int_\Sigma 2HX'(0)\cdot\nu \, d\Sigma \,,$$

and the first variation of the enclosed algebraic volume is given by

$$\mathcal{V}'(0) = \int_\Sigma X'(0)\cdot\nu \, d\Sigma \,. \tag{2}$$

Hence, $H \equiv$constant $\neq 0$ means exactly that the first variation of area vanishes for all volume preserving variations. We remark that the condition

$$\int_\Sigma X'(0)\cdot\nu \, d\Sigma = 0 \tag{3}$$

is known to be sufficient to embed the field $X'(0)$ in a variation $X(\epsilon)$ which fixes the boundary and preserves volume to all orders.

If we assume that (1) preserves the enclosed volume to all orders, the second variation of area is given by

$$\mathcal{A}''(0) = -\int_\Sigma \psi L[\psi] \, d\Sigma =: I[\psi] \,,$$

where $\psi := X'(0)\cdot\nu$ and $L = \Delta + ||d\nu||^2$ is the Jacobi operator [1] . The operator L has a spectrum $\lambda_1 < \lambda_2 \leq \lambda_3 \leq ... \to \infty$ and a corresponding sequence of eigenfunctions $\{\psi_j\}_{j\geq 1}$ satisfying $(L + \lambda_j)[\psi_j] = 0$ in Σ with $\psi_j|_{\partial\Sigma} = 0$. $\{\psi_j\}$ can be chosen so that

[1] Δ is the Laplacian on Σ with the metric induced by X. For the euclidean metric $ds^2 = \sum_{i,j} \delta_{ij} du^i du^j$, $\Delta\varphi = \varphi_{u^1u^1} + \varphi_{u^2u^2}$.

they form an orthonormal basis for $L^2(\Sigma)$ ([7, Lemma 1]). Below we will sometimes write λ_j for $\lambda_j(L,\Sigma)$.

The immersion is called *stable* (sometimes called *weakly stable*) if $\mathcal{A}''(0) \geq 0$ holds for all variations which fix the enclosed volume and keep the boundary fixed. It is clear that stability implies $\lambda_2 \geq 0$ holds (cf. Theorem 2.1). We will call X *critical* if $\lambda_2 = 0$ holds and if

$$\int_\Sigma \psi_2 \, d\Sigma = 0$$

for all eigenfunctions ψ_2 belonging to zero eigenvalue.

For any variation of the surface, the variation of the mean curvature function is given by

$$H'(0) = \frac{1}{2}L[\psi] + \nabla H \cdot X'(0) \, .$$

Of course, if $H \equiv$ constant, then the second term vanishes. We now consider a particular type of variation field which is given by

$$X'(0) = \chi,$$

where χ is a Killing field on $\mathbf{R}^3$. Such fields are given by linear combinations of constant vector fields and fields of the form $\vec{a} \times X$ where $\vec{a}$ is a constant vector field. It is then clear that for such a field, we have

$$0 = H'(0) = \frac{1}{2}L[\chi \cdot \nu]$$

on any CMC surface. We will call the function $\psi := \chi \cdot \nu$ a *Killing-Jacobi field*. A critically stable CMC surface for which ψ_2 is given by a Killing-Jacobi field will be called *Killing critical*. It is this type of surface which will be studied in this paper. This is a special case of (III-B) in Theorem 2.1 below. Examples will be given in sections 5 and 6.

In general, questions about the stability of the surface cannot be answered by only using the λ_j's. The complete story is given by Theorem 2.1 below, which is a refinement of Theorem 1.3 in [4]. Before stating it, we give

Definition 2.1 Let $X : \Sigma \to \mathbf{R}^3$ be a CMC immersion. A smooth variation $X(\epsilon)$ of X is said to be admissible if $X(\epsilon)|_{\partial\Sigma} = X|_{\partial\Sigma}$ is satisfied for all ϵ. For such a variation, we set $u := X'(0) \cdot \nu$.

(i) The variation $X(\epsilon)$ is said to be nontrivial (resp. trivial) if $u \neq 0$ (resp. $u = 0$).

(ii) X is said to be strictly stable if $\mathcal{A}''(0) > 0$ holds for all nontrivial admissible variations.

(iii) X is said to be unstable if $\mathcal{A}''(0) < 0$ holds for some admissible variation.

Set

$$F_0 := \left\{ u \in C_0^{3+\alpha}(\Sigma) \middle| \int_\Sigma u \, d\Sigma = 0 \right\}.$$

It is known that, for any $u \in F_0$, there exists a volume-preserving admissible variation $X(\epsilon)$ of X such that $X'(0) \cdot \nu = 0$ holds (cf.[1, Lemma 2.4]). Using this fact, it is easy to see the following:

Lemma 2.1 *(i) X is strictly stable if and only if $I[u] > 0$ holds for all $u \in F_0 \setminus \{0\}$.*

(ii) X is stable if and only if $I[u] \geq 0$ holds for all $u \in F_0$.

(iii) X is unstable if and only if $I[u] < 0$ holds for some $u \in F_0$.

Denote by E the eigenspace belonging to zero eigenvalue of L with Dirichlet boundary condition, and by $E^\perp$ its orthogonal compliment with respect to L^2-inner product.

Theorem 2.1 *Let $X : \Sigma \to \mathbf{R}^3$ be a $C^{3+\alpha}$-CMC ($0 < \alpha < 1$) immersion.*

(I) If $\lambda_1 \geq 0$, then X is strictly stable.

(II) If $\lambda_1 < 0 < \lambda_2$, then there exists a uniquely determined function $\varphi \in C_0^{2+\alpha}(\Sigma)$ (smooth, in fact) which satisfies $L\varphi = 1$, and the following statements hold.

(II-1) If $\int_\Sigma \varphi \, d\Sigma > 0$, then X is strictly stable.

(II-2) If $\int_\Sigma \varphi \, d\Sigma = 0$, then X is stable but not strictly stable. $\mathcal{A}''(0) = 0$ for a volume-preserving variation $X(\epsilon)$ if and only if $u = a\varphi$ ($a \in \mathbf{R}$).

(II-3) If $\int_\Sigma \varphi \, d\Sigma < 0$, then X is unstable.

(III) If $\lambda_1 < 0 = \lambda_2$, then the following statements hold:

(III-A) If there exists a λ_2-eigenfunction e which satisfies $\int_\Sigma e \, \mathrm{d}\Sigma \neq 0$, then X is unstable.

(III-B) If $\int_\Sigma e \, d\Sigma = 0$ for any λ_2-eigenfunction e, then there exists a uniquely determined function $\varphi \in E^\perp$ which satisfies $L\varphi = 1$ and $\varphi|_{\partial\Sigma} = 0$. Also, the following statements hold:

(III-B1) *If* $\int_\Sigma \varphi \, d\Sigma > 0$*, then* X *is stable but not strictly stable.* $\mathcal{A}''(0) = 0$ *for a volume-preserving variation* $X(\epsilon)$ *if and only if* $u \in E$*.*

(III-B2) *If* $\int_\Sigma \varphi \, d\Sigma = 0$*, then* X *is stable but not strictly stable.* $\mathcal{A}''(0) = 0$ *for a volume-preserving variation* $X(\epsilon)$ *if and only if* $u = e + a\varphi$ *(*$e \in E$*,* $a \in \mathbf{R}$*).*

(III-B3) *If* $\int_\Sigma \varphi \, d\Sigma < 0$*, then* X *is unstable.*

(IV) *If* $\lambda_2 < 0$*, then* X *is unstable.*

The proof of Theorem 2.1 will be given in Appendix A.

3 Calculation of the higher variation

We consider a volume preserving admissible variation of a CMC surface $X : \Sigma \to \mathbf{R}^3$:

$$X(\epsilon) = X + [\epsilon\psi + (\epsilon^2/2)f + \mathcal{O}(\epsilon^3)]\nu$$

that fixes the boundary values, that is, $X(\epsilon)|_{\partial\Sigma} = X|_{\partial\Sigma}$. It is assumed that ψ is a Killing-Jacobi field vanishing on $\partial\Sigma$. We easily get the following:

$$0 = \mathcal{V}' = \int X' \cdot \nu \, d\Sigma \,,$$

$$0 = \mathcal{V}'' = \int (X' \cdot \nu \, d\Sigma)' \,,$$

$$0 = \mathcal{V}''' = \int (X' \cdot \nu \, d\Sigma)'' \,, \tag{4}$$

$$0 = \mathcal{V}^{(4)} = \int (X' \cdot \nu \, d\Sigma)''' \,. \tag{5}$$

$$\mathcal{A}' = -\int 2HX' \cdot \nu \, d\Sigma \,,$$

$$\mathcal{A}'' = -\int 2H(X' \cdot \nu \, d\Sigma)' - \int 2H'(X' \cdot \nu \, d\Sigma) \,,$$

$$\mathcal{A}''' = -\int 2H(X' \cdot \nu \, d\Sigma)'' - \int 4H'(X' \cdot \nu \, d\Sigma)' - \int 2H''(X' \cdot \nu \, d\Sigma) \,.$$

We first consider the third variation of area for a volume constrained variation. From (4) and the fact that $2H' = L[\psi] = 0$, we get

$$\mathcal{A}''' = -\int 2H''\psi \, d\Sigma \,. \tag{6}$$

Next we compute the fourth variation. We obtain

$$\begin{aligned}\mathcal{A}^{(4)} &= -\int 2H(X'\cdot\nu\, d\Sigma)''' - \int 6H''(X'\cdot\nu\, d\Sigma)' \\ &\quad -\int 6H'(X'\cdot\nu\, d\Sigma)'' - \int 2H'''(X'\cdot\nu\, d\Sigma)\,.\end{aligned}$$

Since the mean curvature H is constant, we get from (5),

$$\mathcal{A}^{(4)} = -\int 6H''(X'\cdot\nu\, d\Sigma)' - \int 6H'(X'\cdot\nu\, d\Sigma)'' - \int 2H'''(X'\cdot\nu\, d\Sigma)\,.$$

Since ψ is a Jacobi field, we get, when $\epsilon = 0$, $0 = 2H' = L[\psi]$, so

$$\mathcal{A}^{(4)}(0) = -\int 6H''(X'\cdot\nu\, d\Sigma)' - \int 2H'''\psi\, d\Sigma\,. \tag{7}$$

We next use that for any ϵ,

$$2H'(\epsilon) = L_\epsilon[X'(\epsilon)\cdot\nu(\epsilon)] + 2(\nabla H)(\epsilon)\cdot X'(\epsilon),$$

and so

$$2H'' = L[X''\cdot\nu] + L[X'\cdot\nu'] + L'[X'\cdot\nu] + 2(\nabla H)'\cdot X' + 2\nabla H\cdot X''\,.$$

If we use that $H \equiv$ constant and $H' = 0$, when $\epsilon = 0$, we arrive at

$$2H'' = L[X''\cdot\nu] + L[X'\cdot\nu'] + L'[X'\cdot\nu] = L[f] + L'[\psi]\,. \tag{8}$$

Since χ generates a one parameter family of isometries, we have for any ϵ

$$0 = L_\epsilon[\psi_\epsilon] + 2\nabla_\epsilon H_\epsilon\cdot\chi_\epsilon\,,$$

so, at $\epsilon = 0$ we have

$$0 = L'[\psi] + L[\psi'] + 2(\nabla' H + \nabla H')\cdot\chi + 2\nabla H\cdot\chi' = L'[\psi] + L[\psi'], \tag{9}$$

since H is constant on Σ. From (8) and (9), we obtain

$$2H'' = L[f - \psi']\,, \tag{10}$$

and so from (7), we get

$$\mathcal{A}^{(4)} = -\int 3L[f-\psi'](f - 2H\psi^2)\, d\Sigma - \int 2H'''\psi\, d\Sigma\,. \tag{11}$$

The first equality in (8) holds for all ϵ, so we obtain

$$2H''' = L''[\psi] + 2L'[f] + L[(X' \cdot \nu)''] \, .$$

Note that for any ϵ, $X'(\epsilon) \cdot \nu(\epsilon) \equiv 0$ along $\partial\Sigma$, so the same holds for $(X' \cdot \nu)''$. We get from (11) that

$$\mathcal{A}^{(4)} = -\int 3L[f - \psi'](f - 2H\psi^2) \, d\Sigma - \int (L''[\psi] + 2L'[f])\psi \, d\Sigma \, . \qquad (12)$$

Note that derivatives of the operator L appear in the calculations given above. We will need to consider how the self-adjointness of L affects its derivatives. We consider two fixed smooth functions g and h on Σ which are independent of ϵ. For any ϵ, we have

$$\int_\Sigma gL_\epsilon[h] - hL_\epsilon[g] \, d\Sigma_\epsilon = \oint_{\partial\Sigma} g *_\epsilon dh - h *_\epsilon dg \, ,$$

where $*_\epsilon$ denotes the Hodge star operator relative to the metric induced by $X(\epsilon)$. Differentiating the last formula gives, when $\epsilon = 0$,

$$\int_\Sigma gL'[h] - hL'[g] \, d\Sigma + \int_\Sigma (gL[h] - hL[g])(-2H\psi) \, d\Sigma = \oint_{\partial\Sigma} g *' dh - h *' dg \, .$$

For a surface in $\mathbf{R}^3$, the Hodge operator acting on 1-forms can be expressed

$$*d\phi(w) = -d\phi(\nu \times w) \, , \phi \in C^1(\Sigma), \quad w \in T\Sigma \, .$$

Differentiating this with respect to ϵ, we obtain

$$*' d\phi(w) = -d\phi(\nu' \times w) = d\phi(\nabla\psi \times w)$$

and so we get

$$\begin{aligned} \int_\Sigma gL'[h] - hL'[g] \, d\Sigma + \int_\Sigma (gL[h] - hL[g])(-2H\psi) \, d\Sigma &= \oint_{\partial\Sigma} g *' dh - h *' dg \\ &= \oint_{\partial\Sigma} gdh(\nabla\psi \times t) - hdg(\nabla\psi \times t) \, , \end{aligned}$$

where t is the unit tangent to Σ. Now since $\psi \equiv 0$ on $\partial\Sigma$, $\nabla\psi$ is in the direction of the co-normal to $\partial\Sigma$. This means that $\nabla\psi \times t$ is normal to the surface and so we arrive at

$$\int_\Sigma gL'[h] - hL'[g] \, d\Sigma + \int_\Sigma (gL[h] - hL[g])(-2H\psi) \, d\Sigma = 0 \, . \qquad (13)$$

Using (13), we get

$$-\int_\Sigma 2\psi L'[f]\,d\Sigma = -\int_\Sigma 2fL'[\psi]\,d\Sigma - \int_\Sigma 4H\psi^2 L[f]\,d\Sigma\,.$$

Using (9), we have

$$-\int_\Sigma 2\psi L'[f]\,d\Sigma = \int_\Sigma 2fL[\psi']\,d\Sigma - \int_\Sigma 4H\psi^2 L[f]\,d\Sigma\,. \tag{14}$$

Combining this with (12), we get

$$\begin{aligned}\mathcal{A}^{(4)} \;=\; & -\int_\Sigma 3fL[f]\,d\Sigma + \int_\Sigma 5fL[\psi']\,d\Sigma + \int_\Sigma 2H\psi^2 L[f]\,d\Sigma \\ & -\int_\Sigma 6H\psi^2 L[\psi']\,d\Sigma - \int_\Sigma \psi L''[\psi]\,d\Sigma\,.\end{aligned} \tag{15}$$

We consider the term involving L'' in (12). At any time ϵ, we have

$$L_\epsilon[\psi_\epsilon] = -2\nabla_\epsilon H_\epsilon \cdot \chi_\epsilon\,,$$

$$L'_\epsilon[\psi_\epsilon] + L_\epsilon[\psi'_\epsilon] = -2(\nabla'_\epsilon H_\epsilon + \nabla_\epsilon H'_\epsilon)\cdot\chi_\epsilon - 2\nabla_\epsilon H_\epsilon \cdot \chi'_\epsilon\,.$$

And so at $\epsilon = 0$

$$L''[\psi] + 2L'[\psi'] + L[\psi''] = -2(\nabla'' H + 2\nabla' H' + \nabla H'')\cdot\chi - 4(\nabla' H + \nabla H')\cdot\chi' - 2\nabla H\cdot\chi''\,.$$

Since $H \equiv$ constant, we get $\nabla H = \nabla' H = \nabla'' H = 0$ and we also have $2H' = L[\psi] = 0$. Applying (10) to replace H'', we get

$$L''[\psi] = -2L'[\psi'] - L[\psi''] - \nabla L[f - \psi']\cdot\chi\,. \tag{16}$$

Using (16), we have

$$-\int_\Sigma \psi L''[\psi]\,d\Sigma = \int_\Sigma 2\psi L'[\psi'] + \psi L[\psi''] + \psi\nabla L[f-\psi']\cdot\chi\,d\Sigma\,. \tag{17}$$

Using that ψ vanishes on $\partial\Sigma$, we obtain from the Divergence Theorem

$$\int_\Sigma \psi\nabla L[f-\psi']\cdot\chi\,d\Sigma = -\int_\Sigma \psi L[f-\psi']\nabla\cdot\chi^T\,d\Sigma - \int_\Sigma L[f-\psi']\nabla\psi\cdot\chi\,d\Sigma\,.$$

We use the formulas

$$\nabla\cdot\chi^T = 2H\psi\,, \quad \nabla\psi\cdot\chi = -\psi'\,. \tag{18}$$

Both formulas are easily verified by using that, in general $\chi(X) = \vec{a} \times X + \vec{b}$ for constant vectors $\vec{a}$ and $\vec{b}$. We then obtain

$$\int_\Sigma \psi \nabla L[f - \psi'] \cdot \chi \, d\Sigma = -\int_\Sigma 2H\psi^2 L[f - \psi'] \, d\Sigma + \int_\Sigma \psi' L[f - \psi'] \, d\Sigma \, .$$

Using this in (17), we get

$$-\int_\Sigma \psi L''[\psi] \, d\Sigma = \int_\Sigma 2\psi L'[\psi'] + \psi L[\psi''] - 2H\psi^2 L[f - \psi'] + \psi' L[f - \psi'] \, d\Sigma \, . \quad (19)$$

From (13), we get, with the help of (9),

$$\int_\Sigma 2\psi L'[\psi'] \, d\Sigma = -\int_\Sigma 2\psi' L[\psi'] + 4H\psi^2 L[\psi'] \, d\Sigma,$$

so that (19) becomes

$$-\int_\Sigma \psi L''[\psi] \, d\Sigma = \int_\Sigma -3\psi' L[\psi'] + \psi L[\psi''] - 2H\psi^2 L[f] + \psi' L[f] \, d\Sigma + \int_\Sigma 6H\psi^2 L[\psi'] \, d\Sigma.$$

Replacing the corresponding term in (15), gives

$$\begin{aligned}
\mathcal{A}^{(4)} &= -\int_\Sigma 3fL[f] \, d\Sigma + \int_\Sigma 5fL[\psi'] \, d\Sigma - \int_\Sigma 3\psi' L[\psi'] \, d\Sigma \\
&\quad + \int_\Sigma \psi' L[f] \, d\Sigma + \int_\Sigma \psi L[\psi''] \, d\Sigma \, . \\
&= -\int_\Sigma 3fL[f] \, d\Sigma + \int_\Sigma 6fL[\psi'] \, d\Sigma - \int_\Sigma 3\psi' L[\psi'] \, d\Sigma \\
&\quad + \oint_{\partial\Sigma} \psi' \partial_n f - \psi'' \partial_n \psi \, d\ell \, , \qquad (20)
\end{aligned}$$

where ∂_n is the differential with respect to the outward pointing conormal along $\partial\Sigma$, and we have used integration by parts.

We will show that the boundary integral vanishes. First of all, we have

$$\psi'' = \chi'' \cdot \nu + 2\chi' \cdot \nu' + \chi \cdot \nu'' \, .$$

As mentioned above, the most general form for χ is $\chi = a \times X + b$ for constant vectors a and b. So $\chi' = a \times X' = a \times \psi\nu$ and $\chi'' = a \times X'' = a \times f\nu$, both of which vanish on $\partial\Sigma$.

The second variation of the normal is given by

$$\nu'' = 2\psi d\nu(\nabla\psi) - |\nabla\psi|^2 \nu - \nabla f \, .$$

Since $\psi \equiv 0$ on $\partial\Sigma$ and $\nu \cdot \chi =: \psi$, this gives

$$\nu'' \cdot \chi = -\nabla f \cdot \chi$$

on $\partial\Sigma$. Using that $\psi' = -\nabla\psi \cdot \chi$, the boundary integral becomes

$$\begin{aligned} \oint_{\partial\Sigma} \psi' \partial_n f - \psi'' \partial_n \psi \, d\ell &= \oint_{\partial\Sigma} -(\nabla\psi \cdot \chi)\partial_n f + (\nabla f \cdot \chi)\partial_n \psi \, d\ell \\ &= \oint_{\partial\Sigma} \chi \times (\nabla\psi \times \nabla f) \cdot n \, d\ell \\ &= 0, \end{aligned} \tag{21}$$

since ∇f and $\nabla\psi$ are parallel along the boundary. Above, we have used the triple vector product formula $A \times (B \times C) = (A \cdot C)B - (A \cdot B)C$ for $A, B, C \in \mathbf{R}^3$.

For a compact CMC surface with smooth boundary, we let $\mathcal{K}(\Sigma)$ denote the vector space of Killing Jacobi fields which vanish on $\partial\Sigma$.

Proposition 3.1 *Let Σ be a compact CMC surface in $\mathbf{R}^3$ with smooth boundary $\partial\Sigma$. For $\psi \in \mathcal{K}(\Sigma)$, let $X + [\epsilon\psi + (\epsilon^2/2)f + ...]\nu$ be a volume preserving admissible variation of X. Then the third variation of area is given by*

$$\oint_{\partial\Sigma} \nabla\psi \cdot \chi \partial_n \psi \, d\ell \, , \tag{22}$$

where $\psi = \chi \cdot \nu$.

Proof. We combine (6), (10) and (18), to get

$$\begin{aligned} \mathcal{A}''' = -\int_\Sigma 2H''\psi \, d\Sigma &= -\int_\Sigma \psi(L[f] - L[\psi']) \, d\Sigma \\ &= \int_\Sigma \psi L[\psi'] \, d\Sigma \\ &= -\oint_{\partial\Sigma} \psi' \partial_n \psi \, d\ell \\ &= \oint_{\partial\Sigma} \nabla\psi \cdot \chi \partial_n \psi \, d\ell \, . \end{aligned}$$

q.e.d.

Remark 3.1 It is clear from the previous proposition that the third variation defines a quadratic operator on $\mathcal{K}(\Sigma)$. As such, we will say that the third variation vanishes on Σ if this bilinear form is identically zero. This is equivalent to

$$0 = \int_\Sigma \psi_1 L[\psi_2'] + \psi_2 L[\psi_1'] \, d\Sigma = 0, \text{ for all } \psi_1, \psi_2 \in \mathcal{K}(\Sigma) \, . \tag{23}$$

Remark 3.2 Note that the variation $X+[\epsilon\psi+(\epsilon^2/2)f+...]\nu$ is not necessarily the same as the one parameter family of surfaces obtained by applying the flow of the infinitesimal isometry to the immersion X. For this reason, $\mathcal{A}'''$ does not necessarily vanish.

Combining (20) with (21) gives

Theorem 3.1 *Let Σ be a compact CMC surface in $\mathbf{R}^3$ with smooth boundary $\partial\Sigma$. Assume that the kernel of the Jacobi operator with Dirichlet boundary condition is exactly $\mathcal{K}(\Sigma)$. We assume that for all volume preserving variations $X+(\epsilon\psi+(\epsilon^2/2)f+...)\nu$, $\psi\in\mathcal{K}(\Sigma)$, the third variation of area vanishes. Then, the fourth variation of area is given by*

$$\mathcal{A}^{(4)} = -\int_\Sigma 3fL[f]\,d\Sigma + \int_\Sigma 6fL[\psi']\,d\Sigma - \int_\Sigma 3\psi' L[\psi']\,d\Sigma\,, \tag{24}$$

where $\psi' = -\nabla\psi\cdot\chi$.

Note that, in particular, the fourth variation of area only depends on the second order variation of X. To see what conditions are imposed on f, we consider

$$0=\mathcal{V}' = \int_\Sigma X'\cdot\nu\,d\Sigma\,,$$

$$\begin{aligned} 0=\mathcal{V}'' &= \int_\Sigma X''\cdot\nu\,d\Sigma + \int_\Sigma X'\cdot\nu'\,d\Sigma + \int_\Sigma X'\cdot\nu\,(d\Sigma)' \\ &= \int_\Sigma f\,d\Sigma - \int_\Sigma 2H\psi^2\,d\Sigma\,. \end{aligned}$$

Each boundary component of Σ is assumed to be a closed curve. This means that we can write the Killing field χ either as a constant vector or it is of the form $\vec{C}\times X$ for a constant vector $\vec{C}$. We consider a closed, bounded domain $U\subset\mathbf{R}^3$ such that $\partial U=\Sigma\cup Q$ with the surface Q disjoint from Σ. By applying the Divergence Theoerem to the vector field χ, we get.

$$\int_\Sigma \psi\,d\Sigma = \int_Q \chi\cdot d\vec{S}_Q\,.$$

If we deform the surface Σ, keeping the boundary fixed, we can regard the surface Q as fixed also. Differentiating this equation with respect to ϵ, gives

$$\int_\Sigma \psi'\,d\Sigma - \int_\Sigma 2H\psi^2\,d\Sigma = 0\,.$$

Combining, we arrive at the condition

$$\int_\Sigma f - \psi' \, d\Sigma = 0 \,, \tag{25}$$

for the second order term of a volume preserving variation.

We now regard the right hand side of (24) as a quadratic functional $\mathcal{F} = \mathcal{F}_\psi$:

$$\mathcal{F}_\psi[f] = -\int_\Sigma 3fL[f] \, d\Sigma + \int_\Sigma 6fL[\psi'] \, d\Sigma - \int_\Sigma 3\psi' L[\psi'] \, d\Sigma \tag{26}$$

for the function f. The competing functions are those which vanish on $\partial\Sigma$ and satisfy (25).

Proposition 3.2 *Let Σ be a compact CMC surface with smooth boundary. Assume that $\lambda_2(L, \Sigma) = 0$, that the kernel of the Jacobi operator is exactly $\mathcal{K}(\Sigma)$ and that the surface satisfies the conditions of case (III-B1) of Theorem 2.1. Also assume that the third variation of area vanishes and if dim $\mathcal{K}(\Sigma) > 1$ we assume the stronger condition*

$$0 = \int_\Sigma \psi_1 L[\psi_2'] \, d\Sigma, \text{ for all } \psi_1, \psi_2 \in \mathcal{K}(\Sigma) \,. \tag{27}$$

Then, the functional $\mathcal{F}_\psi$ achieves a minimum in the class of functions vanishing on $\partial\Sigma$ and satisfying (25). This minimum is achieved by the function f_1 satisfying

$$L[f_1] = L[\psi'] + A \,, \quad f_1|_{\partial\Sigma} \equiv 0 \,, \tag{28}$$

where

$$A := \frac{\oint_{\partial\Sigma} \psi' \partial_n \varphi \, d\ell}{\int_\Sigma \varphi \, d\Sigma} \,, \tag{29}$$

(φ is as in Theorem 2.1) and the minimum value is given by

$$\frac{1}{3}\mathcal{F}_{\psi,\min} = -\oint_{\partial\Sigma} \psi' \partial_n f_1 \, d\ell \,. \tag{30}$$

Remark 3.3 By the formula (10), equation (28) is equivalent to the equality $2H''(0) = A$. Since A is constant on Σ, the equation (28) is exactly the condition that the mean curvatures of the surfaces $X(\epsilon) = X + [\epsilon\psi + (\epsilon^2/2)f_1 + ...]\nu$ are constant on Σ (not necessarily constant in ϵ) to second order.

Proof of Proposition 3.2. By the Fredholm Alternative, (28) is solvable provided

$$0 = \int_\Sigma \psi_1 (L[\psi'] + A) \, d\Sigma, \text{ for all } \psi_1 \in \mathcal{K}(\Sigma) \,.$$

If dim $\mathcal{K}(\Psi) = 1$, this condition holds since the integral of every Jacobi field must vanish by the conditions of case (III-B1) of Theorem 2.1 and the hypothesis that the third variation of area vanishes, so by Green's Formula

$$0 = -\int_{\partial\Sigma} \psi' \partial_n \psi \, d\ell = \int_\Sigma \psi L[\psi'] \, d\Sigma = \int_\Sigma \psi (L[\psi'] + A) \, d\Sigma.$$

If dim $\mathcal{K}(\Psi) > 1$ holds, we again have that the integrals of Jacobi fields must vanish by the conditions of (III-B1), but in this case, we use the condition (27).

The equation is therefore solvable for every $A \in \mathbf{R}$, however there is a unique value of A for which the solution will satisfy the volume preserving condition (25). To see this, we multiply (28) by φ and integrate

$$\begin{aligned} 0 &= \int_\Sigma \varphi (L[f_1] - L[\psi'] - A) \, d\Sigma \\ &= \int_\Sigma f_1 - \psi' - A\varphi \, d\Sigma + \oint_{\partial\Sigma} \psi' \partial_n \varphi \, d\Sigma \\ &= -\int_\Sigma A\varphi \, d\Sigma + \oint_{\partial\Sigma} \psi' \partial_n \varphi \, d\Sigma \, . \end{aligned}$$

This gives the equation (29).

The general smooth function vanishing on $\partial\Sigma$ and satisfying the condition (25) is $f = f_1 + \zeta$, where ζ vanishes on $\partial\Sigma$ and

$$\int_\Sigma \zeta \, d\Sigma = 0 \, .$$

We get

$$\begin{aligned} \mathcal{F}_\psi[f_1 + \zeta] &= \mathcal{F}_\psi[f_1] + 6\int_\Sigma \zeta L[\psi'] \, d\Sigma - 3\int_\Sigma \zeta L[f_1] \, d\Sigma - 3\int_\Sigma f_1 L[\zeta] \, d\Sigma - 3\int_\Sigma \zeta L[\zeta] \, d\Sigma \\ &= \mathcal{F}_\psi[f_1] + 6\int_\Sigma \zeta L[\psi'] \, d\Sigma - 6\int_\Sigma \zeta L[f_1] \, d\Sigma - 3\int_\Sigma \zeta L[\zeta] \, d\Sigma \\ &= \mathcal{F}_\psi[f_1] - 6\int_\Sigma A\zeta \, d\Sigma - 3\int_\Sigma \zeta L[\zeta] \, d\Sigma \\ &\geq \mathcal{F}_\psi[f_1] \, , \end{aligned}$$

by the stability of Σ. This shows that $\mathcal{F}_\psi$ achieves its minimum at any function satisfying the equation (28).

If f_1 is as above, then

$$\mathcal{F}_\psi[f_1] = -3\int_\Sigma \psi' L[\psi'] \, d\Sigma + 6\int_\Sigma f_1 L[\psi'] \, d\Sigma - 3\int_\Sigma f_1 (L[\psi'] + A) \, d\Sigma$$

$$
\begin{aligned}
&= 3\int_\Sigma (f_1-\psi')L[\psi']\,d\Sigma - 3\int_\Sigma Af_1\,d\Sigma \\
&= 3\int_\Sigma \psi' L[(f_1-\psi')]\,d\Sigma + 3\oint_{\partial\Sigma}(f_1-\psi')\partial_n\psi' - \psi'\partial_n(f_1-\psi')\,d\ell - 3\int_\Sigma Af_1\,d\Sigma \\
&= 3\oint_{\partial\Sigma}(f_1-\psi')\partial_n\psi' - \psi'\partial_n(f_1-\psi')\,d\ell \\
&= -3\oint_{\partial\Sigma}\psi'\partial_n f_1\,d\ell\,.
\end{aligned}
$$

q.e.d.

Assuming the conditions of the previous theorem hold, a necessary condition for Σ to be a local minimizer among all surfaces having the same boundary and containing the same volume is that $\mathcal{F}_{\psi,min} \geq 0$ holds for all $\psi \in \mathcal{K}(\Sigma)$, while $\mathcal{F}_{\psi,min} > 0$ for all $\psi \in \mathcal{K}(\Sigma)$ is sufficient for local minimization in the C^1 topology.

The following lemma shows that any variation which is volume preserving to third order can be embedded in a genuine volume preserving variation. The proof is a modification of a standard idea.

Lemma 3.1 *Let $X+[\epsilon\psi+(\epsilon^2/2)f+(\epsilon^3/3)g]\nu$ be a variation of X with $0=\partial_\epsilon^k\mathcal{V}=0$, $k=1,2,3$. Then there exists variations of X of the form*

$$X+[\epsilon\psi+(\epsilon^2/2)f+(\epsilon^3/3!)g+\mathcal{O}(\epsilon^4)]\nu \tag{31}$$

with $\mathcal{V}(\epsilon)\equiv$constant.

Proof. We select any function $h \in C_0^\infty(\Sigma)$ with

$$\int_\Sigma h\,d\Sigma \neq 0$$

and consider the two parameter variation

$$X(\epsilon,\sigma) = X+[\epsilon\psi+(\epsilon^2/2)f+(\epsilon^3/3!)g+\sigma h]\nu\,.$$

By assumption, we have

$$\partial_\epsilon\mathcal{V}(0,0)=0\,, \qquad \partial_\sigma\mathcal{V}(0,0)=\int_\Sigma h\,d\Sigma \neq 0\,. \tag{32}$$

By the Implicit Function Theorem, it is possible to find a smooth function $\sigma(\epsilon)$ such that $\sigma(0)=0$ and $\mathcal{V}(\epsilon,\sigma(\epsilon))\equiv$constant. Differentiating the last expression, we get

$$
\begin{aligned}
0 &\equiv \frac{d}{d\epsilon}\mathcal{V}(\epsilon,\sigma(\epsilon)) \\
&= \mathcal{V}_\epsilon(\epsilon,\sigma(\epsilon))+\mathcal{V}_\sigma(\epsilon,\sigma(\epsilon))\sigma'(\epsilon)\,.
\end{aligned}
$$

When $\epsilon = 0$, this gives

$$0 = \mathcal{V}_\epsilon(0,0) + \mathcal{V}_\sigma(0,0)\sigma'(0)\ .$$

Using (32), we obtain $\sigma'(0) = 0$.

Differentiating $\mathcal{V}(\epsilon, \sigma(\epsilon))$ a second time, we obtain

$$\begin{aligned} 0 &\equiv \frac{d^2}{d\epsilon^2}\mathcal{V}(\epsilon,\sigma(\epsilon)) \\ &= \mathcal{V}_{\epsilon\epsilon}(\epsilon,\sigma(\epsilon)) + 2\mathcal{V}_{\epsilon\sigma}(\epsilon,\sigma(\epsilon))\sigma'(\epsilon) + \mathcal{V}_{\sigma\sigma}(\epsilon,\sigma(\epsilon))(\sigma'(\epsilon))^2 + \mathcal{V}_\sigma(\epsilon,\sigma(\epsilon))\sigma''(\epsilon) \end{aligned}$$

Plugging in $\epsilon = 0$ and using the assumption $\mathcal{V}_{\epsilon\epsilon}(0,0) = 0$, (32) and $\sigma'(0) = 0$, we obtain $\sigma''(0) = 0$.

Finally, differentiating $\mathcal{V}(\epsilon, \sigma(\epsilon))$ a third time, we get

$$\begin{aligned} 0 &\equiv \frac{d^3}{d\epsilon^3}\mathcal{V}(\epsilon,\sigma(\epsilon)) \\ &= \mathcal{V}_{\epsilon\epsilon\epsilon}(\epsilon,\sigma(\epsilon)) + 3\mathcal{V}_{\epsilon\epsilon\sigma}(\epsilon,\sigma(\epsilon))\sigma'(\epsilon) + 3\mathcal{V}_{\epsilon\sigma\sigma}(\epsilon,\sigma(\epsilon))(\sigma'(\epsilon))^2 \\ &\quad +3\mathcal{V}_{\sigma\sigma}(\epsilon,\sigma(\epsilon))\sigma'(\epsilon)\sigma''(\epsilon) + \mathcal{V}_{\sigma\sigma\sigma}(\epsilon,\sigma(\epsilon))(\sigma'(\epsilon))^3 + 3\mathcal{V}_{\sigma\epsilon}(\epsilon,\sigma(\epsilon))\sigma''(\epsilon) + \mathcal{V}_\sigma(\epsilon,\sigma(\epsilon))\sigma'''(\epsilon)\ . \end{aligned}$$

Again plugging in $\epsilon = 0$ and using the assumptions of the lemma, (32) and the vanishing of $\sigma'(0)$ and $\sigma''(0)$, we obtain $\sigma'''(0) = 0$. **q.e.d.**

4 Conjugate Delaunay Surfaces

In this section we present material that will be essential in considering the examples which follow.

We identify $\mathbf{R}^2$ with $\mathbf{C}$ and represent a Delaunay surface by

$$X(s,\vartheta) := (r(s)e^{i\vartheta}, z(s))\ ,$$

where s is the arc length parameter of the generating curve (r, z). The corresponding outward pointing normal is given by

$$\nu := (\dot{z}e^{i\vartheta}, -\dot{r}) =: (\cos(\theta)e^{i\vartheta}, \sin\theta))\ , \tag{33}$$

where "dot" means differentiation with respect to s. Using Noether's Theorem, one derives that the condition for X to have constant mean curvature, is equivalent to an equation

$$\cos\theta\, r + Hr^2 - F/4 \equiv 0\ , \tag{34}$$

where F is a constant which we call the *flux parameter* of the surface. Here H denotes the mean curvature, which is non positive by the choice of normal. We refer the reader to [5] for details.

We assume now that $H \neq 0$. Since (34) is a quadratic, it can be solved, yielding

$$r = \frac{\cos\theta \pm \sqrt{\cos^2\theta + HF}}{-2H} .$$

From equation (33), we obtain $dz/dr = -\cot\theta$, so $dz = -\cot\theta \, r_\theta \, d\theta$ which yields using (34),

$$z = \frac{1}{-2H} \int_{\theta_0}^{\theta} (1 \pm \frac{\cos\hat{\theta}}{\sqrt{\cos^2\hat{\theta} + HF}}) \cos\hat{\theta} \, d\,\hat{\theta} \,. \tag{35}$$

For $F > 0$, the surface X parameterizes an unduloid $\mathcal{U}$. In this case, starting from $\theta_0 = 0$, we can take the $+$ branch of the square root until the value $\theta_i = \pm\arccos(\sqrt{-HF})$ is reached. These points correspond to inflection points on the generating curve of $\mathcal{U}$. At these values of θ, we must switch to the negative branch of the root and then continue from θ_i back to $\theta = 0$.

The extremal values of r will be referred to as a bulge and a neck:

$$B := \frac{1 + \sqrt{1 + HF}}{-2H} \,, \; N := \frac{1 - \sqrt{1 + HF}}{-2H} \,.$$

For $F < 0$, the surface is a nodoid $\mathcal{N}$. In this case, we need only to consider the positive branch of the root.

For reasons that will be apparent in the next paragraph, it is interesting to note that equation (34) is a special case of the more general equation which can be used to describe both rotationally invariant and helicoidally invariant surfaces with constant mean curvature, i.e. all CMC surfaces which admit a continuous group of isometries.

Bonnet showed that every simply connected non-spherical CMC surface Σ admits a 2π periodic isometric deformation which preserves the principal curvatures. This deformation is achieved by introducing a one parameter family of second fundamental forms $\{II_\alpha\}_{\alpha\in[0,2\pi)}$ which, together with the original first fundamental form, satisfy the Gauss and Codazzi equations and thus allow the construction of a one parameter family of surfaces over any simply connected domain. The resulted surfaces are referred to as the *associated family* of Σ. The associate family is known to contain all CMC surfaces isometric to a given one that have the same mean curvature. If $\Sigma_0 := \Sigma$, then the surface Σ_π is known as the *conjugate* surface. In particular, when Σ_0 is an unduloid, the conjugate surface is part of a nodoid and all other surfaces in the associated family are invariant under a helicoidal group action. Pictures of this family of surfaces can be found in [3] and [8].

Proposition 4.1 *Let $\mathcal{U}$ be an unduloid with mean curvature H and flux parameter F. Let $a := 1/\sqrt{1+HF}$ and define*

$$\tilde{r} := ar\,, \qquad \tilde{z} := \int_0^s \pm\sqrt{1-(a\dot{r})^2}\,ds\,, \qquad \tilde{\vartheta} := \vartheta/a\,. \tag{36}$$

Then the surface defined by

$$\tilde{X}(s,\vartheta) := (\tilde{r}e^{i\tilde{\vartheta}}, \tilde{z})$$

is the conjugate nodoid to $\mathcal{U}$ and its flux parameter is given by

$$\tilde{F} = -a^2 F\,.$$

In other words

$$\tilde{r}\dot{\tilde{z}} + H\tilde{r}^2 + a^2F/4 = 0 \tag{37}$$

holds.

Remark 4.1 When computing the $\tilde{z}$ coordinate, starting from $s = 0$, the $+$ sign is chosen for the radical until the value of s corresponding to the first inflection point on the generating curve of the unduloid is reached whence the sign is flipped. At the next inflection the sign flips again and so forth.

Proof of Proposition 4.1. It is straightforward to check that the metrics of both X and $\tilde{X}$ are given by $ds^2 + r^2d\vartheta^2$.

With $\tilde{z}$ defined as above, we have

$$(\dot{\tilde{z}})^2/a^2 - (\dot{z})^2 = a^{-2} - 1 \tag{38}$$

holds. The equation (37) is the same as

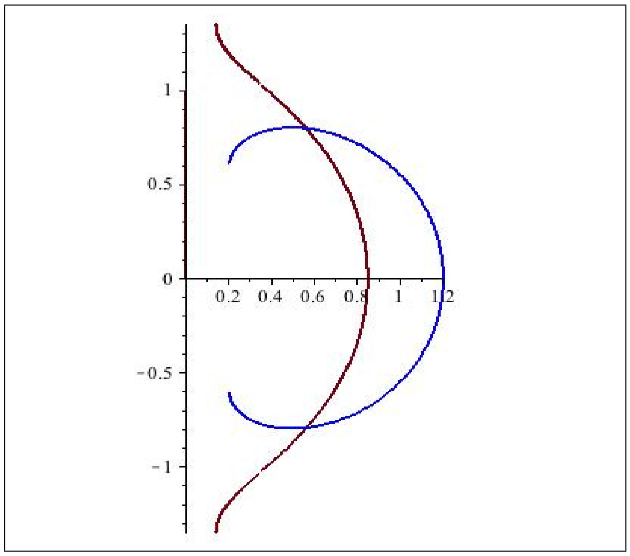
$$(\dot{\tilde{z}})^2/a^2 = (\frac{F}{4r} + Hr)^2. \tag{39}$$

And (34) gives $(\dot{z})^2 = (F/(4r) - Hr)^2$. Then (39) follows since

$$(\frac{F}{4r} + Hr)^2 - (\frac{F}{4r} - Hr)^2 = FH = a^{-2} - 1\,. \tag{40}$$

Since both surfaces also have constant mean curvature H, they are in the same associate family. **q.e.d.**

Figure 1: The generating curve of one period of an unduloid (red) and the generating curve of the corresponding part of its isometric nodoid (blue).



5 Applications to the unduloid

In this section $\mathcal{U}$ will denote one period of an unduloid bounded by two consecutive necks and $\mathcal{N}$ will denote the corresponding domain in the conjugate nodoid. The generating curves for both domains are shown in Figure 1.

Proposition 5.1 *For $\mathcal{U}$, $\lambda_1 < 0 = \lambda_2$ holds. Moreover, the function ν_3 spans the eigenspace belonging to* 0.

Proof. By using the Stone-Weierstrass Theorem, we can reduce the study of the spectrum of L to considering functions of the form $\psi(s,\vartheta) = u(s)\cos(n\vartheta)$ or $\psi(s,\vartheta) = u(s)\sin(n\vartheta), n = 0, 1, 2, ...$ We will assume the ψ has the former form, the case of sin being very similar. We have

$$\begin{aligned} L[\psi] &= \frac{1}{r}(r\psi_s)_s + \frac{1}{r^2}\psi_{\vartheta\vartheta} + (4H^2 - 2K)\psi \\ &= (\frac{1}{r}(ru_s)_s + (4H^2 - 2K - \frac{n^2}{r^2})u)\cos(n\vartheta) \\ &=: (\hat{L} - \frac{n^2}{r^2})[u]\cos(n\vartheta)\,. \end{aligned}$$

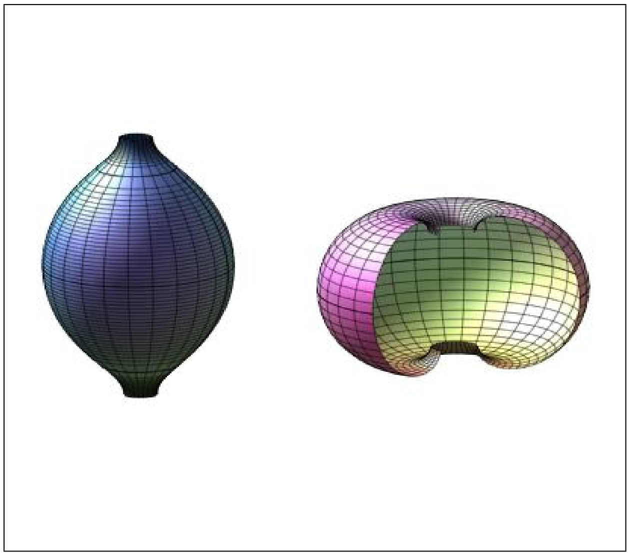


Figure 2: The unduloid on the left, minus a meridian, is isometric to the nodoid domain displayed on the right.

The first component of the Gauss map is given by $\nu_1 = \dot{z}\cos\theta$ and so $(\hat{L} - \frac{1}{r^2})[\dot{z}] = 0$ holds. Note that $w := \dot{z}$ is *positive* on $\mathcal{U}$, since the height function z is strictly increasing along the generating curve.

By using separation of variables, we see that each Dirichlet eigenvalue of L occurs as an eigenvalue of a one dimensional problem:

$$(\hat{L} - \frac{n^2}{r^2} + \mu)[u(s)] = 0\ ,\ \ u(0) = 0 = u(\ell)\ , \tag{41}$$

where ℓ is the length of the generating curve of $\mathcal{U}$.

Note that the first eigenvalue $\hat{\lambda}_1(n)$ of (41) satisfies

$$\hat{\lambda}_1(n) = \min_{u \in C_0^\infty([0,\ell]) - \{0\}} \frac{-\int_0^\ell u(\hat{L} - \frac{n^2}{r^2})[u] r\, ds}{\int_0^\ell u^2 r\, ds}.$$

Let $\zeta = \zeta(s)$ denote any smooth function vanishing at 0 and ℓ and let $n \geq 1$. We have by standard calculations

$$-\int_0^\ell \zeta w(\hat{L} - \frac{n^2}{r^2})[\zeta w]\, rds \ \ = \ \ \int_0^\ell w^2 \zeta_s^2\, rds - \int_0^\ell \zeta^2 w(\hat{L} - \frac{n^2}{r^2})[w]\, rds$$

$$
\begin{aligned}
&= \int_0^\ell w^2 \zeta_s^2 \, r ds - \int_0^\ell \zeta^2 w (\hat{L} - \frac{1}{r^2})[w] \, r ds + \int_0^\ell w^2 \zeta^2 \frac{n^2-1}{r^2} \, r ds \\
&= \int_0^\ell w^2 \zeta_s^2 \, r ds + \int_0^\ell w^2 \zeta^2 (\frac{n^2-1}{r^2}) \, r ds \\
&\geq 0 \, .
\end{aligned}
$$

This means that for $n \geq 1$, all eigenvalues of the problems (41) are non-negative and we need only consider the case $n = 0$ which is the case of axially symmetric variations.

The function $\nu_3 = -\dot{r}(s)$ satisfies $\hat{L}[\nu_3] = 0$ and has exactly two nodal domains, which means that the second eigenvalue of $\hat{L}$ on $[0, \ell]$ is 0. This follows from general results on Sturm-Liouville problems [2]. The first eigenvalue must then be negative since the inequality $\lambda_1 < \lambda_2$ is known to also holds for Sturm-Liouville problems. **q.e.d.**

Lemma 5.1 *Let $X : \Sigma \to \mathbf{R}^3$ be a CMC surface and let $q := X \cdot \nu$ be its support function. Then*

$$L[q] = -2H \tag{42}$$

holds.

Proof. If $X(\epsilon) = X + \epsilon \delta X + \mathcal{O}(\epsilon^2)$ is any variation of X, then the first variation of the mean curvature is

$$\delta H = \frac{1}{2} L[\delta X \cdot \nu] + \delta X \cdot \nabla H \, . \tag{43}$$

We consider the variation by rescalings $X(\epsilon) = (1+\epsilon)^{-1} X$. Then, the mean curvature H_ϵ of $X(\epsilon)$ is $(1+\epsilon)H$. Since $H \equiv$ constant, by using (43), we get (42). **q.e.d.**

Lemma 5.2 *Let $\tilde{q}$ be the support function of $\mathcal{N}$ which may be considered as a function on $\mathcal{U}$ and define*

$$\eta(s) := \tilde{q} + aq \, ,$$

where $a := 1/\sqrt{1 + HF}$ as in Proposition 4.1. Then

$$L[\eta] = -2H(1 + a) > 0$$

and $\eta \equiv 0$ on $\partial \mathcal{U}$.

Proof. Let s_N denote the first positive value of s at which a neck of $\mathcal{U}$ occurs. Then the first neck on $\mathcal{N}$ also occurs at $s = s_N$. To see this, note that the domain in

$\mathcal{U}$ given by $0 < s < s_N$ is a nodal domain for the axially symmetric function ν_3 and so the corresponding domain in $\mathcal{N}$ must also be a nodal domain for an axially symmetric eigenfunction of L. The only axially symmetric eigenfunctions on $\mathcal{N}$ vanishing at $s = 0$ are multiples of $\tilde{\nu}_3$ and their nodal domains correspond to the part of $\mathcal{N}$ between a bulge and a neck.

We have

$$q = r\dot{z} - z\dot{r}\ , \quad \tilde{q} = \tilde{r}\dot{\tilde{z}} - \dot{\tilde{r}}\tilde{z} = a(r(\pm\sqrt{1-(a\dot{r})^2}\,) - \tilde{z}\dot{r})\ .$$

We have

$$\tilde{q}(0) = aB, \quad \tilde{q}(s_N) = -aN.$$

The minus sign occurs since there is exactly one inflection point between 0 and s_N. Also,

$$q(0) = B, \qquad q(s_N) = N.$$

Let $\eta := aq + \tilde{q}$. Then $L[\eta] = aL[q] + L[\tilde{q}] = -2H(1+a) > 0$ and $\eta(s_N) = 0$. **q.e.d.**

Theorem 5.1 *The surface $\mathcal{U}$ is stable in the sense that its second variation of area is non negative for all volume preserving variations. More precisely, (III-B1) in Theorem 2.1 holds.*

Proof. Because of Proposition 5.1 we are in case (III-B) of Theorem 2.1. Note that η is even. Since the function η is a positive multiple of the function φ in Theorem 2.1, we should consider

$$\int_{\mathcal{U}} \eta\, d\Sigma = -2H(1+a)\int_{\mathcal{U}} \varphi\, d\Sigma\ . \tag{44}$$

By applying the Divergence Theorem to the vector field X, we get

$$3\mathcal{V} = \int_{\mathcal{U}} q\, d\Sigma + 2\pi N^2 z(s_N)\ .$$

And so

$$\int_{\mathcal{U}} q\, d\Sigma = 3\mathcal{V} - 2\pi N^2 z(s_N)\ . \tag{45}$$

Similarly, we get

$$\int_{\mathcal{U}} \tilde{q}\, d\Sigma = \int_{\mathcal{U}} \tilde{q}\, d\tilde{\Sigma} = 3\tilde{\mathcal{V}} - 2\pi a^2 N^2 \tilde{z}(s_N). \tag{46}$$

The expressions on the right hand sides of (45) and (46) are both positive since they represent three times the volume of the respective surface minus the volume of the cylinder with radius equal to the neck size and having the same height as the surface.

It follows that the integral of $\eta = aq + \tilde{q}$ is positive since $a > 0$ holds. **q.e.d.**

Remark 5.1 This result was used in [6] in order to study a specific bifurcation of the unduloid.

Although the unduloid $\mathcal{U}$ is stable, it is unclear whether or not it minimizes area for the prescribed volume and boundary because of the existence of a Jacobi field; namely the function ν_3. To explore this question in greater depth, we need to consider higher variations of the area.

Let $X(\epsilon) = X + (\epsilon\nu_3 + (1/2)\epsilon^2 f + \mathcal{O}(\epsilon^3))\nu$ be a volume preserving variation with the same boundary conditions as X. We express the Gauss map of $X(\epsilon)$ as $\nu_\epsilon = \nu + \epsilon\nu' + \mathcal{O}(\epsilon^2)$. ('Prime' denotes differentiation with respect to ϵ.) Since $X' = \nu_3\nu$, we have

$$\nu' = -\nabla\nu_3 \ , \quad \nu_3' = -\nabla\nu_3 \cdot E_3 = -\nabla\nu_3 \cdot \nabla z = \ddot{r}\dot{z} = k_1(\dot{z})^2 \ . \tag{47}$$

Here and afterwards, we will use k_i, $i = 1, 2$ to denote the principal curvatures of the considered surface.

Proposition 5.2 *For variations as above, the third variation of area vanishes.*

Proof. By Proposition 3.1, we must show

$$\oint_{\partial\mathcal{U}} \nabla\nu_3 \cdot E_3 \partial_n \nu_3 \, d\ell = 0 \ . \tag{48}$$

On both components of $\partial\mathcal{U}$, we get $\nabla\nu_3 \cdot E_3 = \nabla\nu_3 \cdot \nabla x_3 = k_1$ which is an even function. On the other hand $\partial_n\nu_3 = -\ddot{r} = -k_1$ on the upper boundary component and $\partial_n\nu_3 = +\ddot{r} = +k_1$ on the lower boundary component and therefore the integral in (48) vanishes. **q.e.d.**

Theorem 5.2 *For the Killing critical domain of the unduloid, let $X(\epsilon) = X + [\epsilon\nu_3 + (1/2)\epsilon^2 f_1 + \mathcal{O}(\epsilon^3)]\nu$ be the volume preserving variation constructed in the proof of Proposition 3.2 with $\psi = \nu_3$. Then the fourth variation of area is negative.*

Proof. In this case, we can explicitly find the solution of

$$L[f_1] = L[\nu_3'] + A \ , \quad f_1|_{\partial\mathcal{U}} \equiv 0 \ , \tag{49}$$

where

$$A := \frac{\oint_{\partial\mathcal{U}} \nu_3' \partial_n \varphi \, d\ell}{\int_{\mathcal{U}} \varphi \, d\Sigma} \ . \tag{50}$$

Let $j = j(s)$ denote the even solution of

$$\hat{L}[j] = 0\,, \quad j \equiv 1 \text{ on } \partial\mathcal{U}\,.$$

This problem (49) has a solution since ν_3 vanishes on $\partial\mathcal{U}$ and j and ν_3 cannot simultaneously vanish. Then using (47), we get that

$$f_1 := \nu_3' - (2H + 1/N)j + A\varphi\,, \tag{51}$$

since $\nu_3' = k_1 = 2H - k_2 = 2H + 1/N$ on $\partial\mathcal{U}$. The general solution is obtained by adding an arbitrary multiple of the function ν_3.

It follows easily from the calculations in the proof of Lemma 5.2, that

$$\varphi = \frac{\tilde{q} + aq}{-2H(1+a)}\,, \quad j := \frac{q - \tilde{q}}{N(1+a)}\,. \tag{52}$$

So by (51), the function f_1 is more or less explicitly known. Moreover, on $\partial\mathcal{U}$, we have

$$\partial_n \nu_3' = \pm\partial_s[k_1(\dot{z})^2]_{s=s_N} = \pm[\partial_s(2H - k_2)(\dot{z})^2]_{s=s_N} = \pm\partial_s[(2H + \dot{z}/r)(\dot{z})^2]_{s=s_N} = 0,$$

since both $\dot{z}$ and r have critical points at $s = s_N$. Therefore we have

$$\begin{aligned}
\mathcal{F}_{\min} = \mathcal{F}[f_1] &= -3\oint_{\partial\mathcal{U}} \nu_3' \partial_n f_1\, d\ell \\
&= -3(2H + 1/N)\oint_{\partial\mathcal{U}} \partial_n f_1\, d\ell \\
&= -3(2H + 1/N)\oint_{\partial\mathcal{U}} \partial_n[\nu_3' - (2H + 1/N)j + A\varphi]\, d\ell \\
&= -3(2H + 1/N)\oint_{\partial\mathcal{U}} \partial_n[-(2H + 1/N)j + A\varphi]\, d\ell \\
&= \frac{3(2H + 1/N)^2}{\int_{\mathcal{U}} \varphi\, d\Sigma}\left(\int_{\mathcal{U}} \varphi\, d\Sigma \oint_{\partial\mathcal{U}} \partial_n j\, d\ell - [\oint_{\partial\mathcal{U}} \partial_n \varphi\, d\ell]^2\right) \\
&= \frac{3(2H + 1/N)^2}{\int_{\mathcal{U}} \varphi\, d\Sigma}\left(\int_{\mathcal{U}} \varphi\, d\Sigma \oint_{\partial\mathcal{U}} \partial_n j\, d\ell - \oint_{\partial\mathcal{U}} \partial_n \varphi\, d\ell \int_{\mathcal{U}} j\, d\Sigma\right).
\end{aligned}$$

Below, we use $(\cdot)_n$ to denote differentiation with respect to the outward pointing unit normal along $\partial\mathcal{U}$. Using (52), we get

$$\begin{aligned}
\mathcal{F}_{\min} &= \frac{3(2H + 1/N)^2}{-2HN(1+a)^2 \int_{\mathcal{U}} \varphi\, d\Sigma}\left(\oint_{\partial\mathcal{U}} q_n - \tilde{q}_n\, d\ell \int_{\mathcal{U}} \tilde{q} + aq\, d\Sigma - \oint_{\partial\mathcal{U}} \tilde{q}_n + aq_n\, d\ell \int_{\mathcal{U}} q - \tilde{q}\, d\Sigma\right) \\
&= \frac{3(2H + 1/N)^2}{-2HN(1+a) \int_{\mathcal{U}} \varphi\, d\Sigma}\left(\oint_{\partial\mathcal{U}} q_n\, d\ell \int_{\mathcal{U}} \tilde{q}\, d\Sigma - \oint_{\partial\mathcal{U}} \tilde{q}_n\, d\ell \int_{\mathcal{U}} q\, d\Sigma\right).
\end{aligned}$$

Finally, we use that $q_n(s_N) = -(2H + 1/N)z(s_N)$ (< 0), $\tilde{q}_n(s_N) = -(2H + 1/\tilde{N})\tilde{z}(s_N)$ (> 0) , (45) and (46) to get

$$\begin{aligned} \mathcal{F}_{\min} &= \frac{3(2H+1/N)^2 \mathrm{Length}[\partial\mathcal{U}]}{-2HN(1+a)\int_{\mathcal{U}} \varphi \, d\Sigma} \Big(-(2H+1/N)z(s_N)[3\tilde{\mathcal{V}} - 2\pi \tilde{N}^2 \tilde{z}(s_N)] \\ &\quad + (2H+1/\tilde{N})\tilde{z}(s_N)[3\mathcal{V} - 2\pi N^2 z(s_N)] \Big) < 0 \, . \end{aligned}$$

The last inequality follows since $3\mathcal{V} - 2\pi N^2 z(s_N)$, (resp. $3\tilde{\mathcal{V}} - 2\pi \tilde{N}^2 \tilde{z}(s_N)$) is three times the volume enclosed by $\mathcal{U}$ (resp. $\mathcal{N}$) minus the volume enclosed in the cylinder having the same boundary, so both of these quantities are positive.

6 Applications to the nodoid

In this section, we show that a stable, Killing critical domain of a particular nodoid is area minimizing in comparison to nearby surfaces sharing the same boundary and enclosed volume.

Let $\mathcal{U}$ denote an unduloid domain generated by the part of an undulary curve between two inflection points which contains a bulge. Note that the surface $\mathcal{U}$ differs from the surface represented by the same letter in the previous section. Up to rigid motions in $\mathbf{R}^3$, there is a two parameter family of such domains and we fix one of these parameters by requiring $H = -1$. We fix the second parameter by taking the flux parameter $F = 3/4$. By the results of section 4, this unduloid is locally isometric to a nodoid domain $\mathcal{N}$. If we represent $\mathcal{U}$ as the image of an immersion

$$X = (re^{i\vartheta}, z) \, ,$$

then $\mathcal{N}$ is represented

$$\tilde{X} = (\tilde{r}e^{i\tilde{\vartheta}}, \tilde{z}) := (2re^{i\vartheta/2}, \int_0^s \sqrt{1 - 4\dot{r}^2} \, ds) \, .$$

We wish to emphasize that $\mathcal{U}$ is only isometric to $\mathcal{N}$ locally. In this case, we have $a = 2$. The inflection points on the generating curve of $\mathcal{U}$ occur when

$$0 = \sqrt{\cos^2\theta + HF} = \sqrt{\cos^2\theta - 3/4} \, ,$$

where $\dot{z} = \cos\theta$, $-\dot{r} = \sin\theta$. So at the inflection points, we have $\dot{\tilde{z}} = \sqrt{1 - (2(1/2))^2} = 0$, which means that the nodoid $\mathcal{N}$ lies between two circles of points having horizontal tangent planes. Also note that since $\tilde{\vartheta} = \vartheta/2$, the unduloid $\mathcal{U}$, minus a generating curve, is isometric to half of a nodoid as shown in Figure 3.

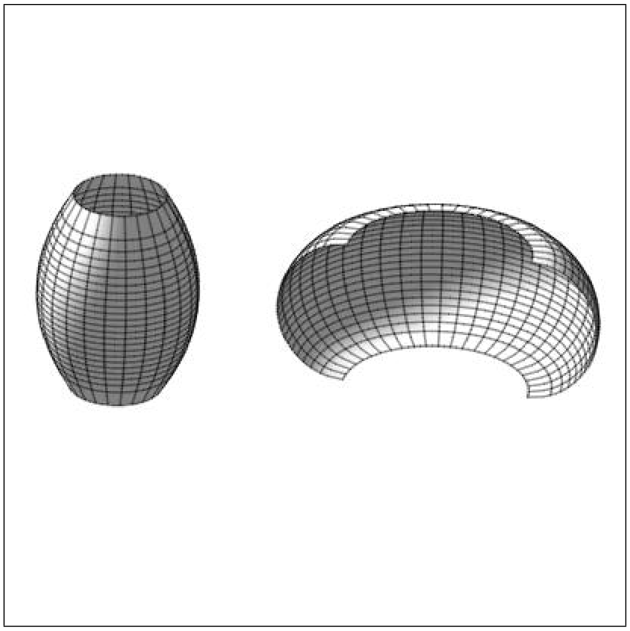


Figure 3: The unduloid $\mathcal{U}$ and the isometric subdomain $\mathcal{N}_1 \subset \mathcal{N}$.

Being locally isometric and having the same mean curvature, the unduloid and the nodoid share the same Jacobi operator which is given by

$$L[\psi] = \frac{1}{r}(r\psi_s)_s + \frac{1}{r^2}\psi_{\vartheta\vartheta} + (4H^2 - 2K)\psi \, . \tag{53}$$

Lemma 6.1 $\mathcal{N}$ *is a Killing critical domain.*

Proof. Since the tangent planes are horizontal along both circles comprising the boundary of $\mathcal{N}$ are horizontal, the functions $\tilde{\nu}_i$, $i = 1, 2$ are Killing-Jacobi fields for $\mathcal{N}$. Since these functions both change sign, we get $\tilde{\lambda}_1 < 0$, $\tilde{\lambda}_2 \leq 0$. We need to show that $\tilde{\lambda}_2 = 0$.

Firstly, there is no negative eigenvalue belonging to an axially symmetric function except for $\tilde{\lambda}_1$. To see this, let w denote the even axially symmetric Jacobi field on the nodoid normalized by $w(0) = 1$, $w_s(0) = 0$, where s is the arc length along the generating curve measured from the bulge. The function w must have a sign change since $\tilde{\lambda}_1 < 0$ holds. Since $\tilde{\nu}_3$ is the odd Jacobi field, we get

$$\tilde{r}(\tilde{\nu}_3 w_s - w\tilde{\nu}_{3_s}) \equiv \text{constant} = \tilde{k}_1 \tilde{r} \tilde{z}_s(0) < 0 \, .$$

Let $s = s_0$ denote the arclength from the bulge to upper boundary circle. On the interval $(0, s_0)$, both $\tilde{\nu}_3$ and $\tilde{\nu}_{3_s}$ are positive. Then it follows from the previous equation that w cannot increase on any interval in $(0, s_0)$ on which w is negative. So w has exactly one nodal domain in $(-s_0, s_0)$.

Let $f(s)$ be an eigenfunction belonging to the second eigenvalue of L restricted to axially symmetric functions. By the Sturm Oscillation Theorem, f has exactly two nodal domains and if the eigenvalue is negative, ν_3 must change sign in each of these nodal domains. This gives a contradiction since ν_3 has only one sign change.

It is well known that the eigenvalue problem for (53) with Dirichlet boundary conditions can be reduced to separation of variables. Writing $\psi(s, \vartheta) = f(s)e^{in\vartheta}$, the equation for f becomes

$$\frac{1}{r}(rf_s)_s + (||d\nu||^2 - \frac{n^2}{r^2} + \lambda)f = 0\,. \tag{54}$$

We have established that there is exactly one negative eigenvalue in the case $n = 0$. When $n = 1$, the function $\tilde{z}_s$ is a non negative solution of (54) with $\lambda = 0$ which vanishes on the boundary so there can be no negative eigenvalues occurring for $n \geq 1$. **q.e.d.**

Proposition 6.1 *$\mathcal{N}$ is stable. More precisely, (III-B1) in Theorem 2.1 holds.*

Proof. The eigenfunctions $\tilde{\nu}_i$, $i = 1, 2$ satisfy

$$\int_{\mathcal{N}} \tilde{\nu}_i \, d\Sigma = 0,$$

so we are in case (III-B) of Theorem 2.1. Since $H = -1$ on $\mathcal{N}$, the solution of $L[\varphi] = 1$ is given by

$$\varphi := \frac{1}{2}(\tilde{q} - \tilde{z}_T\eta_0)\,, \tag{55}$$

where $\tilde{z}_T$ is the height on the top circle of $\mathcal{N}$ and $\eta_0 = \eta_0(s)$ is the even Jacobi field which is identically 1 on this circle. This is because, $\tilde{q}$ satisfies $L[\tilde{q}] = -2H$ and the boundary values of $\tilde{q}$ are equal to $\tilde{z}_T$.

We have

$$\begin{aligned} 2\int_{\mathcal{N}} \varphi \, d\Sigma &= \int_{\mathcal{N}} \tilde{q}\, d\Sigma - \tilde{z}_T \int_{\mathcal{N}} \eta_0 \, d\Sigma \\ &= 3\mathcal{V}(\mathcal{N}) - 2\pi\tilde{r}_T^2\tilde{z}_T - \frac{1}{2}\tilde{z}_T \int_{\mathcal{N}} \eta_0 L[\tilde{q}]\, d\Sigma \\ &= 3\mathcal{V}(\mathcal{N}) - 2\pi\tilde{r}_T^2\tilde{z}_T - \frac{1}{2}\tilde{z}_T \oint_{\partial\mathcal{N}} \partial_n\tilde{q} - \tilde{q}\partial_n\eta_0 \, d\ell \end{aligned}$$

$$= \quad 3\mathcal{V}(\mathcal{N}) - 2\pi\tilde{r}_T^2\tilde{z}_T + \frac{1}{2}\tilde{z}_T^2\oint_{\partial\mathcal{N}}\partial_n\eta_0\,d\ell - \frac{1}{2}\tilde{z}_T\oint_{\partial\mathcal{N}}\partial_n\tilde{q}\,d\ell.$$

The term $3\mathcal{V}(\mathcal{N}) - 2\pi\tilde{r}_T^2\tilde{z}_T$ represents three times the volume of $\mathcal{N}$ minus the volume of the cylinder having the same boundary. This is clearly positive. Above, we showed that even Jacobi field $w(s)$, normalized by $w(0) = 1$, must be negative and decreasing on $\partial\mathcal{N}$. However $\eta_0(s) = w(s)/w(s_0)$ which is increasing on $\partial\mathcal{N}$. This shows that the third term is non negative. On $\partial\mathcal{N}$, $\partial_n\tilde{q} = -2\tilde{r}_T$, so the last term is non negative as well. **q.e.d.**

Remark 6.1 For the nodoid domain we are considering, $\mathcal{K}(\mathcal{N})$ =span $\{\tilde{\nu}_1, \tilde{\nu}_2\}$. For any $\psi \in \mathcal{K}(\mathcal{N})$, we can write, by a standard manipulation

$$\psi = a\tilde{\nu}_1 + b\tilde{\nu}_2 = \tilde{z}_s(a\cos\tilde{\vartheta} + b\sin\tilde{\vartheta}) = \sqrt{a^2+b^2}\,\tilde{z}_s\cos(\tilde{\vartheta} - \delta),$$

for a constant δ. After linear change of the the angle $\tilde{\vartheta}' := \tilde{\vartheta} - \delta$, we arrive at $\psi = \sqrt{a^2+b^2}\,\tilde{z}_s\cos(\tilde{\vartheta}')$. For this reason, it is enough to only consider the function $\tilde{\nu}_1$.

Lemma 6.2 *For any volume preserving variation of the nodoid $\mathcal{N}$ satisfying $\mathcal{A}''(0) = 0$, the third variation of area is zero.*

Proof. By the remark above, it is sufficient to consider variations with $\tilde{X}(0)\cdot\tilde{\nu} = \tilde{\nu}_1$. For $E_1 = (1, 0, 0)$,

$$\begin{aligned}
\tilde{\nu}_1' &= -\nabla\tilde{\nu}_1\cdot E_1 = -\nabla\tilde{\nu}_1\cdot\nabla\tilde{x}_1 \\
&= -(\tilde{r}_s\cos\tilde{\vartheta}\tilde{X}_s - \frac{\tilde{r}\sin\tilde{\vartheta}}{\tilde{r}^2}\tilde{X}_\vartheta)\cdot(\tilde{z}_s\cos\vartheta X_s - \frac{\tilde{z}_s}{\tilde{r}^2}\sin\tilde{\vartheta}\tilde{X}_\vartheta) \\
&= -(\tilde{r}_s\tilde{z}_{ss}\cos^2\tilde{\vartheta} + \frac{\tilde{z}_s}{\tilde{r}}\sin^2\tilde{\vartheta}) \\
&= \tilde{k}_1\tilde{\nu}_3^2\cos^2\tilde{\vartheta} + \tilde{k}_2\sin^2\tilde{\vartheta}\,. \qquad (56)
\end{aligned}$$

On $\partial\mathcal{N}$, $\tilde{k}_2 = 0$ and $\tilde{\nu}_3 = \pm 1$, so

$$\tilde{\nu}_1' = 2H\cos^2\tilde{\vartheta} = H(1 + \cos 2\tilde{\vartheta}) \text{ on } \partial\mathcal{N}. \qquad (57)$$

Denote by $\tilde{C}_+$ (resp. $\tilde{C}_-$) the top (resp. bottom) circle of $\partial\mathcal{N}$. Then, by Proposition 3.1 and (57), we get,

$$\begin{aligned}
\mathcal{A}''' &= \oint_{\partial\mathcal{N}} \nabla\tilde{\nu}_1 \cdot E_1 \partial_n \tilde{\nu}_1 \, d\ell \\
&= H \oint_{\tilde{C}_+} (1+\cos(2\tilde{\vartheta}))\tilde{z}_{ss}(s_0)\cos\tilde{\vartheta}\, d\ell - H \oint_{\tilde{C}_-} (1+\cos(2\tilde{\vartheta}))\tilde{z}_{ss}(-s_0)\cos\tilde{\vartheta}\, d\ell \\
&= H \int_0^{2\pi} (1+\cos(2\tilde{\vartheta}))\tilde{z}_{ss}(s_0)\cos\tilde{\vartheta}\, \tilde{r} d\tilde{\vartheta} - H \int_0^{2\pi} (1+\cos(2\tilde{\vartheta}))\tilde{z}_{ss}(s_0)\cos\tilde{\vartheta}\, \tilde{r} d\tilde{\vartheta} \\
&= 0\,.
\end{aligned}$$

q.e.d.

Theorem 6.1 *The minimum of the fourth variation of area of the nodoid domain $\mathcal{N}$, restricted to volume preserving variations, is positive.*

Proof. In this case, we first take the Killing-Jacobi field to be the function $\tilde{\nu}_1$. The metric on $\mathcal{N}$ is $dS^2 = ds^2 + r^2 d\vartheta^2$. Let η_k, $k = 0, 1, 2, \ldots$ denote the even solution of

$$\frac{1}{\tilde{r}}(\tilde{r}\eta_s)_s + (||d\tilde{\nu}||^2 - k^2/\tilde{r}^2)\eta = 0\,,$$

normalized so that $\eta_k \equiv 1$ on $\partial\mathcal{N}$. It follows that $\eta_k \cos k\tilde{\vartheta}$ is a Jacobi field on $\mathcal{N}$ and hence it is also a Jacobi field on $\mathcal{U}$. The solution φ of $L[\varphi] = 1$ with zero boundary values is

$$\varphi = \frac{q - c\eta_0}{2}\,, \tag{58}$$

where c is the value of q on the top circle. Using (57), we get that the function defined by

$$f_1 = \tilde{\nu}_1' - H(\eta_0(s) + \eta_2(s)\cos(2\tilde{\vartheta})) + A\varphi\,, \tag{59}$$

where

$$A := \frac{\oint_{\partial\mathcal{N}} \tilde{\nu}_1' \partial_n \varphi \, d\ell}{\int_{\mathcal{N}} \varphi \, d\Sigma}\,, \tag{60}$$

satisfies the equation $L[f_1] = L[\nu_1'] + A$ and $f_1|_{\partial\mathcal{N}} = 0$. Using Remark 6.1 and the Fredholm Alternative, we then get that the condition (27) holds and by Proposition 3.2,

$$\frac{1}{3}\mathcal{F}_{\tilde{\nu}_1,\min} = -\oint_{\partial\mathcal{N}} \tilde{\nu}_1' \partial_n f_1 \, d\ell\,, \tag{61}$$

which we now compute.

Note that $\tilde{\nu}_1'$ is an even function of s and so $\partial_s \tilde{\nu}_1'$ is odd and $\partial_n \tilde{\nu}_1' = \pm \partial_s \tilde{\nu}_1'$ is even. Using (56), we get on the top circle

$$\begin{aligned}
\partial_n \tilde{\nu}_1' &= \partial_s \tilde{\nu}_1' \\
&= \tilde{k}_{1s} \cos^2 \tilde{\vartheta} + \tilde{k}_{2s} \sin^2 \tilde{\vartheta} \\
&= -\tilde{k}_{2s} \cos(2\tilde{\vartheta}) \,.
\end{aligned}$$

We compute

$$-\tilde{k}_{2s} = \partial_s(\frac{\tilde{z}_s}{\tilde{r}}) = \frac{\tilde{r}\tilde{z}_{ss} - \tilde{r}_s \tilde{z}_s}{\tilde{r}^2} = \frac{\tilde{k}_1}{\tilde{r}_T} = \frac{2H}{\tilde{r}_T} \,.$$

Therefore

$$\partial_n f_1 = \frac{2H}{\tilde{r}_T} \cos 2\tilde{\vartheta} - H(\eta_{0n} + \eta_{2n} \cos(2\tilde{\vartheta})) + A\varphi_n \,. \tag{62}$$

We claim that for the particular nodoid that we have chosen, $\partial_n \eta_2 = 0$ holds. As mentioned above, $\eta_2(s)\cos(2\tilde{\vartheta})$ is a Jacobi field on both $\mathcal{N}$ and $\mathcal{U}$ which is even with respect to s. However $\tilde{\vartheta} = \vartheta/2$ so this function can be expressed as $\eta_2(s)\cos\vartheta$ on $\mathcal{U}$. But, up to a scalar multiple, the only even Jacobi field on $\mathcal{U}$ of this form is $\nu_1 = z_s \cos(\vartheta)$. On the boundary of $\mathcal{U}$, we have $\partial_s z_s = -k_1 r_s = 0$ since the points on the boundary of $\mathcal{U}$ are inflection points of the generating curve. This proves the claim.

By (57), (59) and (62), we have

$$\begin{aligned}
\frac{1}{3}\mathcal{F}_{\min} := \frac{1}{3}\mathcal{F}_{\tilde{\nu}_1,\min} &= -H \oint_{\partial\mathcal{N}} (1 + \cos 2\tilde{\vartheta})(\frac{2H}{\tilde{r}_T} \cos 2\tilde{\vartheta} - H\eta_{0n} + A\varphi_n)\, d\ell \\
&= \oint_{\partial\mathcal{N}} \eta_{0n} + A\varphi_n d\ell - 4\pi \,,
\end{aligned} \tag{63}$$

where we have used that $H = -1$, $d\ell = \tilde{r}_T\, d\tilde{\vartheta}$ and that the integral over $\partial\mathcal{N}$ includes the integrals over both the top and bottom circles.

Using (57) and (60), we get

$$\frac{1}{3}\mathcal{F}_{\min} = \frac{1}{\int_{\mathcal{N}} \varphi\, d\Sigma}\Big(\oint_{\partial\mathcal{N}} \eta_{0n} d\ell \int_{\mathcal{N}} \varphi\, d\Sigma - (\oint_{\partial\mathcal{N}} \varphi_n\, d\ell)^2 - 4\pi \int_{\mathcal{N}} \varphi\, d\Sigma\Big) \,. \tag{64}$$

We have shown above that the first factor on the right hand side is positive. Note that

$$\oint_{\partial\mathcal{N}} \varphi_n\, d\ell = \oint_{\partial\mathcal{N}} \eta_0 \varphi_n\, d\ell = \int_{\mathcal{N}} \eta_0 \,,$$

since $L[\varphi]=1$ and $L[\eta_0]=0$. Using this to replace one of the factors above, we get

$$\frac{1}{3}\mathcal{F}_{\min} = \frac{1}{\int_{\mathcal{N}} \varphi \, d\Sigma}\Big[\oint_{\partial\mathcal{N}} \eta_{0n} d\ell \int_{\mathcal{N}} \varphi \, d\Sigma - \oint_{\partial\mathcal{N}} \varphi_n \, d\ell \int_{\mathcal{N}} \eta_0 \, d\Sigma - 4\pi \int_{\mathcal{N}} \varphi \, d\Sigma\Big] \, . \tag{65}$$

Note that the first two terms in the square brackets form a determinant. Using (58) we first get

$$\frac{1}{3}\mathcal{F}_{\min} = \frac{1}{\int_{\mathcal{N}} \varphi \, d\Sigma}\Big[\frac{1}{2}\oint_{\partial\mathcal{N}} \eta_{0n} d\ell \int_{\mathcal{N}} q \, d\Sigma - \frac{1}{2}\oint_{\partial\mathcal{N}} q_n \, d\ell \int_{\mathcal{N}} \eta_0 \, d\Sigma - 4\pi \int_{\mathcal{N}} \varphi \, d\Sigma\Big] \, . \tag{66}$$

Secondly, we use that

$$\eta_0 = \frac{\tilde{q}-q}{(\tilde{q}-q)|_{\partial\mathcal{N}}} \, , \tag{67}$$

which gives

$$\frac{1}{3}\mathcal{F}_{\min} = \frac{1}{\int_{\mathcal{N}} \varphi \, d\Sigma}\Big[\frac{1}{2(\tilde{q}-q)|_{\partial\mathcal{N}}}\big(\oint_{\partial\mathcal{N}} \tilde{q}_n \, d\ell \int_{\mathcal{N}} q \, d\Sigma - \oint_{\partial\mathcal{N}} q_n \, d\ell \int_{\mathcal{N}} \tilde{q} \, d\Sigma\big) - 4\pi \int_{\mathcal{N}} \varphi \, d\Sigma\Big] . \tag{68}$$

Denote by C_+ (resp. C_-) the top (resp. bottom) circle of $\partial\mathcal{U}$.

We compute

$$\pm q_n = q_s = r z_{ss} - z r_{ss} = -k_1(r r_s + z z_s) = 0 \quad \text{on } C_\pm \, , \tag{69}$$

since the boundary of $\mathcal{U}$ consists of inflection points of the generating curve. Also, on $\tilde{C}_\pm$,

$$\pm\tilde{q}_n = \tilde{q}_s = \tilde{r}\tilde{z}_{ss} - \tilde{z}\tilde{r}_{ss} = -\tilde{k}_1(\tilde{r}\tilde{r}_s + \tilde{z}\tilde{z}_s) = \pm\tilde{k}_1\tilde{r} = \pm 2H\tilde{r}_T = \mp 2\tilde{r}_T \, . \tag{70}$$

We have

$$\begin{aligned}\int_{\mathcal{N}} \varphi \, d\Sigma &= \frac{1}{2}\int_{\mathcal{N}} \varphi L[q] \, d\Sigma \\ &= \frac{1}{2}\big(\int_{\mathcal{N}} q \, d\Sigma + \oint_{\partial\mathcal{N}} \varphi q_n - q\varphi_n \, d\ell \,\big) \, .\end{aligned}$$

Using (55) and (67) to replace φ and then using (69) and (70) , we get

$$\int_{\mathcal{N}} \varphi \, d\Sigma = \frac{1}{2}\big(\int_{\mathcal{N}} q \, d\Sigma + \frac{1}{2}\big(1 - \frac{\tilde{z}_T}{(\tilde{q}-q)|_{\partial\mathcal{N}}}\big)\oint_{\partial\mathcal{N}} \tilde{q}q_n - q\tilde{q}_n \, d\ell \,\big)$$

$$
\begin{aligned}
&= \frac{1}{2}\Big(\int_{\mathcal{N}} q\, d\Sigma - \frac{1}{2}\big(1 - \frac{\tilde{z}_T}{(\tilde{q}-q)|_{\partial\mathcal{N}}}\big) \oint_{\partial\mathcal{N}} q\tilde{q}_n\, d\ell\ \Big) \\
&= \frac{1}{2}\Big(\int_{\mathcal{N}} q\, d\Sigma - \frac{1}{2}\big(1 - \frac{\tilde{z}_T}{(\tilde{q}-q)|_{\partial\mathcal{N}}}\big)4\pi\tilde{r}_T q_T(-2\tilde{r}_T)\ \Big) \\
&= \frac{1}{2}\Big(\int_{\mathcal{N}} q\, d\Sigma + 4\pi q_T \tilde{r}_T^2\big(1 - \frac{\tilde{z}_T}{(\tilde{q}-q)|_{\partial\mathcal{N}}}\big)\ \Big).
\end{aligned}
$$

Using this in (68) gives,

$$
\frac{1}{3}\mathcal{F}_{\min} = \frac{4\pi}{\int_{\mathcal{N}}\varphi\, d\Sigma}\Big[\big(\frac{-\tilde{r}_T^2}{(\tilde{q}-q)|_{\partial\mathcal{N}}} - \frac{1}{2}\big)\int_{\mathcal{N}} q\, d\Sigma - 2\pi q_T\tilde{r}_T^2\big(1 - \frac{\tilde{z}_T}{(\tilde{q}-q)|_{\partial\mathcal{N}}}\big)\Big]\,. \tag{71}
$$

We will numerically verify that this quantity is positive.

Using the formulas in Section 4, we get

$$
r_T = \sqrt{3}/4,\ \ \tilde{r}_T = \sqrt{3}/2 = (z_s)_T\,.
$$

This gives,

$$
q_T = \frac{\sqrt{3}}{4}\cdot\frac{\sqrt{3}}{2} + \frac{z_T}{2} = \frac{3}{8} + \frac{z_T}{2}\,,
$$

while

$$
\tilde{q}_T = \tilde{z}_T\,.
$$

By (35), we get

$$
z_T = \frac{1}{2}\int_0^{\pi/6}\big(1 + \frac{\cos\hat{\theta}}{\sqrt{\cos^2\hat{\theta} - 3/4}}\big)\cos\hat{\theta}\, d\,\hat{\theta} \approx 0.9837311046\,, \tag{72}
$$

and

$$
\tilde{z}_T = \frac{1}{2}\int_0^{\pi/6}\sqrt{1 - 4\sin(\hat{\theta})^2}\big(1 + \frac{\cos\hat{\theta}}{\sqrt{\cos^2\hat{\theta} - 3/4}}\big)\, d\hat{\theta} \approx 0.7031494430\,.
$$

Using the formulas above, we get

$$
(\tilde{q}-q)|_{\partial\mathcal{N}} = \tilde{z}_T - \frac{z_T}{2} - \frac{3}{8} \approx -.1637161092\,,
$$

and

$$
\frac{-\tilde{r}_T^2}{(\tilde{q}-q)|_{\partial\mathcal{N}}} - \frac{1}{2} \approx 4.081100810\,.
$$

In the second term of (71), the integral of q over $\mathcal{N}$ is equal to $2(3\mathcal{V}(\mathcal{U}) - \mathcal{V}(C))$, where C is the cylinder having the same boundary as $\mathcal{U}$. The factor 2 is because $\mathcal{N}$ is a double cover of $\mathcal{U}$ up to a set of zero measure. Using the formulas given at the start of section 4, we arrive at

$$\int_{\mathcal{N}} q \, d\Sigma \approx 13.32232153 \, .$$

Combining theses calculations as in (71), we obtain

$$\frac{1}{4\pi} \int_{\mathcal{N}} \varphi \, d\Sigma (\frac{1}{3} \mathcal{F}_{\tilde{\nu}_1, \min}) \approx 32.73990212 \, .$$

We have shown above that the integral appearing on the right hand side is positive so the positivity of $\mathcal{F}_{\tilde{\nu}_1, \min}$ follows. We remark that the numerical values obtained above were evaluated as elliptic integrals using Maple.

By Remark 6.1, $\mathcal{F}_{\psi, \min} > 0$ holds for all $\psi \in \mathcal{K}(\mathcal{N})$. **q.e.d.**

A Proof of Theorem 2.1

(I) and (IV) are the same as (I) and (IV) in Theorem 1.3 in [4].

Assume that $\lambda_1 < 0$. Let ψ_i be the eigenfunction corresponding to the i-th eigenvalue λ_i of the problem

$$L\psi = -\lambda\psi, \quad \psi \in H_0^1 - \{0\}, \tag{73}$$

where

$$L = \Delta + ||d\nu||^2.$$

We choose $\{\psi_i\}$ so that they form an orthonormal basis for $L^2(\Sigma)$. Since ψ_1 does not change sign,

$$\int_{\Sigma} \psi_1 \, d\Sigma \neq 0. \tag{74}$$

For a function $u \in C_0^{2+\alpha}(\Sigma)$, set

$$v = -\frac{\int_{\Sigma} u \, d\Sigma}{\int_{\Sigma} \psi_1 \, d\Sigma} \psi_1 + u. \tag{75}$$

Then

$$\int_{\Sigma} v \, d\Sigma = 0. \tag{76}$$

By setting

$$a = -\frac{\int_\Sigma u \, d\Sigma}{\int_\Sigma \psi_1 \, d\Sigma},$$

v is represented as

$$v = a\psi_1 + u.$$

When (73) has no zero eigenvalue, by the Fredholm alternative, there exists a unique function $u \in C_0^{2+\alpha}(\Sigma)$ satisfying $Lu = 1$. In this case, by using the Green's formula and (76), we see that

$$\begin{aligned} I[v] &= -\int_\Sigma vLv \, d\Sigma \\ &= -\int_\Sigma (a^2\psi_1 L\psi_1 + a\psi_1 Lu + auL\psi_1 + uLu) \, d\Sigma \\ &= a^2\lambda_1 \int_\Sigma \psi_1^2 \, d\Sigma - 2a \int_\Sigma \psi_1 Lu \, d\Sigma - \int_\Sigma u \, d\Sigma \\ &= a^2\lambda_1 + \int_\Sigma u \, d\Sigma. \end{aligned} \tag{77}$$

Since $\lambda_1 < 0$, $I[v] < 0$ holds if $\int_\Sigma u \, d\Sigma < 0$ holds. In this case, in view of Lemma 2.1, X is unstable, which proves (II-3).

Next, when $u = \psi_2$, from (77) and the orthonormality of $\{\psi_i\}$, we see

$$\begin{aligned} I[v] &= a^2\lambda_1 \int_\Sigma \psi_1^2 \, d\Sigma - 2a \int_\Sigma \psi_1 L\psi_2 \, d\Sigma - \int_\Sigma \psi_2 L\psi_2 \, d\Sigma \\ &= a^2\lambda_1 \int_\Sigma \psi_1^2 \, d\Sigma + 2a\lambda_2 \int_\Sigma \psi_1\psi_2 \, d\Sigma + \lambda_2 \int_\Sigma \psi_2^2 \, d\Sigma \\ &= \lambda_1 \Big(\int_\Sigma \psi_2 \, d\Sigma\Big)^2 \Big(\int_\Sigma \psi_1 \, d\Sigma\Big)^{-2} + \lambda_2. \end{aligned}$$

Therefore, if $\lambda_2 \le 0$ and $\int_M \psi_2 \ne 0$, then $I[v] < 0$ and so X is unstable, which proves (III-A).

Next we prove (II-1). Set

$$E_1 = \{a\psi_1 | a \in \mathbf{R}\}, \qquad E_1^\perp = \Big\{u \in C_0^{2+\alpha}(\Sigma) \Big| \int_\Sigma \psi_1 u \, d\Sigma = 0\Big\}.$$

Again by the Fredholm alternative, there exists a unique function $u \in C_0^{2+\alpha}(\Sigma)$ satisfying $Lu = 1$. If

$$\int_\Sigma u \, d\Sigma \ge 0,$$

then

$$I[u] = -\int_\Sigma uLu \; d\Sigma = -\int_\Sigma u \; d\Sigma \le 0. \tag{78}$$

Note

$$\begin{aligned}
\lambda_1 = I[\psi_1] &= \min\Big\{ I[u] \;\Big|\; u \in H_0^1(\Sigma) \text{ and } \int_\Sigma u^2 \; d\Sigma = 1 \Big\}, \\
\lambda_i = I[\psi_i] &= \min\Big\{ I[u] \;\Big|\; u \in H_0^1(\Sigma),\ \int_\Sigma u^2 \; d\Sigma = 1 \\
&\qquad \text{and } \int_\Sigma u\psi_j \; d\Sigma = 0 \text{ for } j \in \{1, \cdots, i-1\} \Big\}, \quad i = 2, 3, \cdots.
\end{aligned} \tag{79}$$

(78), (79) and $\lambda_2 > 0$ imply $u \notin E_1^\perp$. Therefore, any $v \in F_0$ is represented as follows:

$$v = w + bu, \quad b \in \mathbf{R}, \quad w \in E_1^\perp.$$

Then

$$\begin{aligned}
I[v] &= -\int_\Sigma (b^2 uLu + buLw + bwLu + wLw) \; d\Sigma \\
&= -b^2 \int_\Sigma u \; d\Sigma - 2b \int_\Sigma w \; d\Sigma + I[w] \\
&= b^2 \int_\Sigma u \; d\Sigma - 2b \int_\Sigma (w + bu) \; d\Sigma + I[w] \\
&= -b^2 I[u] + I[w].
\end{aligned}$$

Note $I[w] > 0$ if $w \neq 0$. Hence, $I[v] \ge 0$ holds, which implies that X is stable. If $\int_\Sigma u \; d\Sigma > 0$, then X is strictly stable, which proves (II-1). If $\int_\Sigma u \; d\Sigma = 0$, then $I[v] = 0$ if and only if $v = bu$ ($b \in \mathbf{R}$), which proves (II-2).

Lastly, we prove (III-B). Again by the Fredholm alternative, there exists a unique function $u \in E^\perp \cap C_0^{2+\alpha}(\Sigma)$ that satisfies $Lu = 1$.

When $\int_\Sigma u \; d\Sigma < 0$, we can prove that X is unstable by the same way as in the proof of (II-2). (III-B1) is proved by the same way as the proof of (II-1). When $\int_\Sigma u \; d\Sigma = 0$,

$$I[u] = -\int_\Sigma uLu \; d\Sigma = -\int_\Sigma u \; d\Sigma = 0 = \lambda_2.$$

Assume that $u \in E_1^\perp$. By (79), u is an eigenfunction corresponding to $\lambda_2 = 0$, that is, $Lu = 0$, which is a contradiction. Therefore, $u \notin E_1^\perp$, and hence, we can prove the stability of X by the same way as in the proof of (II-2). **q.e.d.**

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