# REMARKS ON A CONVERGENT FLOW FOR MARGINALLY TRAPPED SURFACES IN LORENTZ-MINKOWSKI SPACE

BY BENNETT PALMER

ABSTRACT. We study a flow for a class of space-like marginally trapped surfaces in Lorentz-Minkowski space under which the surface converges to a space-like surface with zero mean curvature. The flow decreases the area monotonically.

An analogous flow is constructed for marginally trapped surfaces in an arbitrary four dimensional Lorentzian manifold which satisfies the Null Convergence Condition.

## 1. INTRODUCTION

One of the central problems in the modern calculus of variations is to understand the evolution of non-equilibrium surface for a variational problem to an equilibrium one. Primarily in the Riemannian case, mean curvature and related flows have been applied in a variety of settings to understand this process. This note will concentrate on the somewhat unique situation of a certain class of space-like marginally trapped surfaces in four dimensional Lorentz-Minkowski space evolving to a zero mean curvature surface. What makes this problem unique, and somewhat simple, is the existence of a gauge for which it reduces to a linear problem.

Recall that a surface $\Sigma$ in a Lorentzian 4-manifold is called *marginally trapped* if its mean curvature vector defines a future (or past) pointing null section of the normal bundle. Marginally trapped surface were formally introduced by Penrose [7] in the study of space-time singularities. It is worth pointing out however that they arose in a natural way earlier in the work of Blaschke [3] with regards to conformal and Laguerre differential geometry.

At each point of a space-like surface in $\mathbf{R}^4_1$, the normal space to the surface contains exactly two null vectors of the form $(\nu_\pm, 1)$ with $\nu_\pm \in S^2$. We call the surface a *spherical graph* if either of these sections defines an injective map into the 2-sphere. If, say $\nu_+$ is injective, $\nu_+(\Sigma) =: \Omega$ and the mean curvature $\vec{H}$ is parallel to $(\nu_+, 1)$, we will call $\Sigma$ a *marginally trapped spherical graph over* $\Omega$. In this case, the surface can be described as the image of a *conformal* parameterization $(\nu_+)^{-1} =: Y : \Omega \to \mathbf{R}^4_1$. The class of such surface turns out to be huge, any smooth function on a domain in $S^2$ generates an example. The principal objective of this paper is to show that any marginally trapped spherical graph over $\Omega$ can be evolved, through an area decreasing flow, to a zero mean curvature surface which is also a spherical graph over $\Omega$.

In our earlier paper [5], it was shown that the zero mean curvature spherical graphs over a domain $\Omega$ are the absolute minimizers of area among all marginally trapped spherical graphs over $\Omega$ having the same boundary values up to first order. It is our hope that the result presented here can be extended to marginally trapped surfaces in some other Lorentzian four-manifolds. Some evidence for this lies in the result of [1], where it is shown that in any Lorentzian four-manifold satisfying the null convergence condition, the space-like zero mean curvature surfaces are stable with respect to variations through marginally trapped surfaces fixing the boundary to first order. Towards this end, we formulate a flow for a space like marginally trapped surface in a Lorentzian 4-manifold which satisfies the Null Convergence Condition, which preserves the property of being marginally trapped and decreases area monotonically.

What sparked out interest in the result presented here is a recent paper of Guilfoyle and Klingenberg [4]. In their paper, a convex surface $S \subset \mathbf{R}^3$ is embedded in the space of lines in $\mathbf{R}^3$ via its normal line congruence. The space of lines in $\mathbf{R}^3$ can be identified with $TS^2$, the tangent bundle of $S^2$, which carries a pseudo-Kähler structure of signature $(2,2)$ and, with respect to this structure, the normal line congruence of a surface is Lagrangian. They show that this surface evolves to one with zero mean curvature under Lagrangian mean curvature flow. The flow used in [4] is closely related to the flow we employ here but is distinct from ours in that it is of second order while ours is fourth order. Both flows, however, are expressed in the same gauge, for which they are linear.

## 2. Results.

For any spherical graph over $\Omega$, we define its support function by

$$q := \langle Y, (\nu, 1)\rangle \,, \tag{1}$$

where $\nu$ is the position vector of $S^2$ and $\langle \cdot, \cdot \rangle$ is used for the Lorentzian product in $\mathbf{R}^4_1$. We denote the Laplacian on $S^2$ by $\Delta$. The gradient of a function $u$ on $S^2$ will be denoted $Du$. The following results can be found in [5].

**Lemma 2.1.** *The marginally trapped immersion $Y$ is represented by the formula*

$$Y = (Dq + q\nu, 0) - \frac{1}{2}(\Delta q + 2q)(\nu, 1)\,. \tag{2}$$

*Conversely, if $q \in C^\infty(\Omega)$, then this formula represents a marginally trapped spherical graph over $\Omega$. Further, the immersion $Y : \Omega \to \mathbf{R}^4_1$ is conformal with metric given by*

$$\langle dY, dY\rangle = \frac{1}{4}\big((\Delta q)^2 - 4M[q]\big)\, d\mathcal{S}^2\,, \tag{3}$$

*where $M[\cdot]$ denotes the Monge-Ampere operator and $d\mathcal{S}^2$ denotes the standard metric on $S^2$. The surface defined by $Y$ has zero mean curvature if and only if*

$$\Delta(\Delta + 2)q = 0 \tag{4}$$

*holds.*

Almost all of Lemma 2.1 was proved in [5]. To prove the remaining part, we need the following.

**Lemma 2.2.** *For a smooth function* $f$ *on* $S^2$*, there holds*

(i) $(\Delta Df) \cdot \nu = -2\Delta f$ ,

(ii) $(\Delta Df)^T = D\Delta f$ .

*Proof* We use "$\cdot$" to denote the inner product in $\mathbf{R}^3$. Since $Df \cdot \nu \equiv 0$, there holds $0 = \Delta(Df \cdot \nu) = (\Delta Df) \cdot \nu - 2Df \cdot \nu + 2d\nu \cdot d(Df) = (\Delta Df) \cdot \nu + 2\Delta f$. Here we have used that $\Delta\nu = -2\nu$ and that $d\nu$ is the identity on $TS^2$. This proves (i) since $(dDf)^T = D^2 f$ and $D^2 f \cdot I = 2\Delta f$.

We extend $f$ to a smooth function on a neighborhood of $S^2$. Then $Df = \sum_{j=1}^{3} (\partial_j f) E_j^T$, where $\{E_j\}$ denotes the standard basis of $\mathbf{R}^3$ and the superscript $T$ denotes projection to $TS^2$. Below, we regard $Df$ as a map from $\Omega$ into $\mathbf{R}^3$ and its Laplacian $\Delta Df$ is also regarded as an $\mathbf{R}^3$ valued map.

$$\Delta f = \sum_{k=1}^{3} \big((1 - \nu_k^2)\partial_{kk}^2 f - 2\nu_k \partial_k f\big) . \tag{5}$$

So

$$D\Delta f = \sum_{j,k=1..3} \big((1 - \nu_k^2)\partial_{kkj}^3 f - 2\nu_k \partial_{jk}^2 f - 2\nu_j \partial_{jj}^2 f - 2\partial_j f\big) E_j^T . \tag{6}$$

On the other hand,

$$(\Delta Df)^T = \sum_{j=1}^{3} \big((\Delta\partial_j f) E_j^T + 2(dE_j^T (D\partial_j f))^T + (\partial_j f)(\Delta E_j^T)^T\big) . \tag{7}$$

Replacing $f$ with $\partial_j f$ in (5), we see that the first term in (7) is equal to the first two terms of (6).

Since $E_j^T = E_j - \nu_j \nu$, we have $2(dE_j^T((D\partial_j f))^T = -2\nu_j D(\partial_j f) = -2\nu_j \sum_k (\partial_{jk}^2 f) E_k^T$. Summing this over $j$ and switching the indices gives the third term in (6). Finally, using $\Delta\nu = -2\nu$, we get $(\Delta(E_j^T))^T = (\Delta(E_j - \nu_j \nu))^T = -(\Delta(\nu_j \nu))^T = -2D\nu_j = -2E_j^T$. **q.e.d.**

*Proof of Lemma 2.1.* All of Lemma 2.1 was proved in [5] except for the sufficiency of (4) to imply $\vec{H} \equiv 0$ which we prove now. Using (3), we get

$$2\vec{H}_Y = \Delta_Y Y = \frac{\Delta(\Delta + 2)q}{\frac{1}{4}\big((\Delta q)^2 - 4M[q]\big)} (\nu, 1) . \tag{8}$$

From (2), we have

$$\begin{aligned} \Delta Y &= \big(\Delta Dq + (\Delta q)\nu - 2q\nu, 0\big) - \frac{1}{2}(\Delta(\Delta + 2)q)\big(\nu, 1\big) + ((\Delta + 2)q)\big(\nu, 0\big) - \big(D\Delta q + 2Dq, 0\big) \\ &= -\frac{1}{2}(\Delta(\Delta + 2)q)\big(\nu, 1\big) , \end{aligned}$$

Therefore, for a marginally trapped surface, zero mean curvature is equivalent to (4). **q.e.d.**

Because of (2) and Lemma 2.1, we consider the evolution of the support function given by

$$(\partial_t + \Delta(\Delta+2))u = 0, \ \ u(x,0) = q(x) \tag{9}$$

with the boundary values held constant to first order. The unusual choice of the $+$ sign is because the principal part of the operator is fourth order and has positive eigenvalues. By applying (2) and regarding the functions $u(x,t)$ in (9) as evolving support functions, we obtain a deformation of $Y$ through marginally trapped surfaces.

**Theorem 2.3.** *Let* $Y : \Omega \to \mathbf{R}^4_1$ *be a marginally trapped spherical graph with support function* $q := \langle Y, (\nu, 1)\rangle$*. Then the solution of the evolution problem (9) exists for all* $t > 0$ *and evolves* $Y$ *to a zero mean curvature spherical graph over* $\Omega$*. Throughout the evolution, the boundary of* $Y$ *and the area decreases monotonically and the boundary points are displaced along null geodesics.*

**Lemma 2.4.** *Let* $\Omega \subset S^2$ *be a smoothly bounded domain. For* $\phi \in C^\infty(\bar\Omega)$*, the problem*

$$(\partial_t + \Delta(\Delta+2))u = 0 \, , u(x,0) = \phi \, , \text{in } \Omega \, ,$$
$$u(x,t) = \phi(x), \ \partial_n u(x,t) = \partial_n \phi(x) \text{ on } \partial\Omega$$

*has a smooth solution* $u(x,t)$ *on* $\bar\Omega \times [0,\infty)$.

*In addition ,*

$$\lim_{t\to\infty} u(x,t) = w(x)$$

*holds, where* $w(x)$ *is the unique solution of*

$$\Delta(\Delta+2)w = 0 \, , \text{in } \Omega \, , \tag{10}$$

$$w(x) = \phi(x), \ \partial_n w(x) = \partial_n \phi(x) \text{ on } \partial\Omega \, . \tag{11}$$

*Proof.* In $\Omega$, we consider the spectrum $\mu_1 \leqslant \mu_2 \leqslant \mu_3 \leqslant ...$ for the problem

$$\Delta(\Delta+2)\psi = \mu\psi \, , \text{in } \Omega \, ,$$
$$\psi(x) = 0, \ \partial_n \psi(x) = 0 \text{ on } \partial\Omega \, .$$

The eigenvalues can be produced from the well known max/min principle

$$\mu_j = \sup_{V_j} \inf_{\psi \in V_j^\perp} \frac{\int_\Omega \psi\Delta(\Delta+2)\psi \, d\omega}{\int_\Omega \psi^2 \, d\omega} \, ,$$

where $V_j$ is a $j-1$ dimensional subspace of $W^{2,2}_0$. We normalize the eigenfunctions $\psi_j$ to be orthonormal, with respect to the $L^2$ inner product for the spherical metric. Because of this characterization, all of the eigenvalues decrease monotonically for a monotonically increasing sequence of domains. Taking such a sequence which exhausts $S^2$, we obtain a lower bound

$$\mu_j > \lambda_j^2 - 2\lambda_j \, , \tag{12}$$

where $0 = \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 ...$ is the spectrum of $-\Delta$ on all of $S^2$, the eigenvalues being counted with multiplicity. It is well known that if $\{\lambda'_i\}_{i=0..\infty}$ denote these same eigenvalues without multiplicity, then $\lambda_i' = i(i+1)$ and the multiplicity is $2i+1$. In particular, all of the $\mu_j$'s are positive and $\mu_j \sim \lambda_j^2$ as $j \to \infty$.

Let

$$\beta_1 = \beta_1(\Omega) := \inf_{\psi \in W_0^{2,2}} \frac{\int_\Omega (\Delta\psi)^2 \, d\omega}{\int_\Omega |D\psi|^2 \, d\omega} \geqslant \inf_{\psi \in W_0^{1,2}} \frac{\int_\Omega (\Delta\psi)^2 \, d\omega}{\int_\Omega |D\psi|^2 \, d\omega} = A_1(\Omega) > 2.$$

Then we have, using the definition of $\beta_1$,

$$\begin{aligned} \mu_j &= \mu_j \int_\Omega \psi_j^2 \, d\omega = \int_\Omega \Delta\psi_j (\Delta + 2)\psi_j \, d\omega \\ &= \int_\Omega (\Delta\psi_j)^2 - 2|D\psi_j|^2 \, d\omega \\ &\geqslant (1 - 2/\beta_1) \int_\Omega (\Delta\psi_j)^2 \, d\omega \, . \end{aligned} \tag{13}$$

By the Lichnerowicz formula $(1/2)\Delta|D\zeta|^2 = |D^2\zeta|^2 + |D\zeta|^2 + D\zeta \cdot D\Delta\zeta$ holds for $\zeta \in C_C^\infty(\Omega)$. Integrating this over $\Omega$ gives

$$\int_\Omega (\Delta\zeta)^2 \, d\omega = \int_\Omega |D^2\zeta|^2 + |D\zeta|^2 \, d\omega \, .$$

This equality extends to all $W_0^{2,2}$ functions, so we obtain, using (13), that there exists a constant $c_1 = c_1(\Omega)$ such that for all $j = 1, 2, 3, \ldots$, there holds

$$c_1 \mu_j \geqslant \|\psi_j\|^2_{W^{2,2}(\Omega)} \, ,$$

so by the Sobolev Embedding Theorem, there is a constant $c_2$ with

$$c_2 \mu_j \geqslant \|\psi_j\|_{C^1(\Omega)} \, . \tag{14}$$

We next consider the function formally defined by

$$V(x,t) := \sum_{j=1}^{\infty} e^{-\mu_j t} \hat{\phi}(j) \psi_j(x) \, ,$$

where

$$\hat{\phi}(j) := \int_\Omega \phi(x) \psi_j(x) \, d\omega \, .$$

Note that $\hat{\phi}(j) \to 0$ as $j \to \infty$. By the discussion above, $V \in W^{2,2}(\Omega \times [\epsilon, \infty))$ for all $\epsilon > 0$. Also, it follows from (14), the Weierstrass M-test and the growth rate of the $\mu_j$'s, that $V(x,t)$ is infinitely differentiable in $t$.

Let $V_N$ denote the N'th partial sum of the series defining V and for $\epsilon > 0$, let $\xi(x,t) \in C_c^\infty(\Omega \times (\epsilon, \infty))$. We have

$$\begin{aligned} 0 &= \int_{\Omega\times(\epsilon,\infty)} \xi(\partial_t + \Delta(\Delta+2))[V_N]\, d\omega\, dt \\ &= \int_{\Omega\times(\epsilon,\infty)} V_N(-\partial_t + \Delta(\Delta+2))[\xi]\, d\omega\, dt \\ &\to \int_{\Omega\times(\epsilon,\infty)} V(-\partial_t + \Delta(\Delta+2))[\xi]\, d\omega\, dt \\ &= \int_{\Omega\times(\epsilon,\infty)} \xi V_t + V\Delta(\Delta+2)[\xi]\, d\omega\, dt\,. \end{aligned}$$

From the last line, we can deduce that for all $t > 0$, fixed $V(\cdot,t)$ is a $W_0^{2,2}$ weak solution of $\Delta(\Delta+2)[V] = -V_t$. By elliptic regularity, ([2], Theorems 3.54 and 3.57) and a simple bootstrap argument, V must be a classical solution and by the remarks above about the differentiability with respect to t, V solves

$$(\partial_t + \Delta(\Delta+2))V = 0\,, V(x,0) = \sum_j \hat{\phi}(j)\psi_j(x)\,, \text{in } \Omega\,, \tag{15}$$

$$V(x,t) = 0,\ \partial_n V(x,t) = 0 \text{ on } \partial\Omega\,.$$

Let $w$ be the function as in the statement of the lemma. Note that the boundary value problem (10), (11) is well posed since $\beta_1 > 2$ holds. If we define

$$u(x,t) := w(x) + V(x,t)\,, \tag{16}$$

then we have that $P[u] = 0$ holds. The initial condition holds since, when $t = 0$, the right hand side of (16) is the representation of $\phi$ as the sum of the solution of $\Delta(\Delta+2)w = 0$, $w - \phi \in W_0^{2,2}$ and the solution of $\Delta(\Delta+2)V(\cdot,0) = \Delta(\Delta+2)\phi$ with $V(\cdot,0) \in W_0^{2,2}$. That $u(x,t) \to w(x)$ as $t \to \infty$ follows from the positivity of the $\mu_j'$s. **q.e.d.**

*Proof of Theorem 2.3* The existence of the flow follows from the previous lemma. We now show that the flow (9) decreases area.

By (3), the area of a marginally trapped spherical graph having support function $u$ is

$$\mathcal{A}[u] = \frac{1}{4}\int_\Omega \big((\Delta u)^2 - 4M[u]\big)\, d\omega\,.$$

Using the Lichnerowicz formula, this can be expressed

$$\mathcal{A}[u] = \frac{1}{4}\left(\int_\Omega (\Delta u)(\Delta+2)u\, d\omega + \oint_{\partial\Omega} (\frac{1}{2} * d\|Du\|^2 - \Delta u * du)\right)\,.$$

Let $T$ denote the unit tangent vector to $\partial\Omega$. For the boundary integrand above, we have

$$\begin{aligned}
\frac{1}{2} * d\|Du\|^2 - \Delta u * du &= (D_n Du \cdot Du - (\Delta u) Du \cdot n)\, ds \\
&= \big([(D_n Du \cdot n)(n \cdot Du) + (D_n(Du) \cdot T)(T \cdot Du)] \\
&\quad -[(D_n Du \cdot n) + (D_T Du \cdot T)](Du \cdot n)\big)\, ds \\
&= \big((D_n(Du) \cdot T)(T \cdot Du) - (D_T Du \cdot T)(Du \cdot n)\big)\, ds\,.
\end{aligned}$$

We now assume that $u$ depends on $t$ as in (9). If we differentiate the previous line with respect to $t$, we obtain

$$\big((D_n(Du_t)\cdot T)(T\cdot Du) + (D_n(Du)\cdot T)(T\cdot Du_t) - (D_T Du_t\cdot T)(Du\cdot n) + (D_T Du\cdot T)(Du_t\cdot n)\big) ds.$$

Keeping in mind that the boundary values of $u$ are kept fixed to first order, all the terms clearly vanish except for the first one. However,

$$D_n(Du_t) \cdot T = D_T(Du_t) \cdot n = T(Du_t \cdot n) - Du_t \cdot D_T n = 0\,.$$

Therefore, we have

$$\begin{aligned}
\partial_t A[u(\cdot, t)] &= \frac{1}{4}\int_\Omega \Delta u_t (\Delta + 2) u + \Delta u (\Delta + 2) u_t \, d\omega \\
&= \frac{1}{2}\int_\Omega u_t \Delta(\Delta + 2) u \, d\omega \\
&= -\frac{1}{2}\int_\Omega \big(\Delta(\Delta + 2) u\big)^2 \, d\omega \\
&\leqslant 0\,.
\end{aligned}$$

## 3. Extension of the flow to Lorentzian manifolds

In this section we will formulate a version of the flow given above for surfaces in a general four dimensional Lorentzian manifold satisfying the null convergence condition (NCC) $\mathrm{Ric}_{\mathcal{L}}(Z, Z) \geqslant 0$ for all null vectors $Z$. We will assume the notation of [1].

Let $(\mathcal{L}, \langle\cdot,\cdot\rangle)$ be a four dimensional time oriented Lorentzian manifold and let $Y : \Sigma \to \mathcal{L}$ be a space like surface. Locally, we can frame the normal bundle $\perp Y$ by a pair of null normal fields $\{\vec{a}, \vec{b}\}$ satisfying $\langle \vec{a}, \vec{b} \rangle \equiv 1$. There is a gauge freedom in choosing such a frame given by $\vec{a} \to c\vec{a}$, $\vec{b} \to c^{-1}\vec{b}$, for $c \in \mathbf{R}^*$. We fix this gauge as follows. The time orientability of $\mathcal{L}$ implies the existence of a global future pointing time-like vector field $\vec{\tau}$ on $\mathcal{L}$, satisfying $\langle \vec{\tau}, \vec{\tau} \rangle \equiv -1$, [6], and we require that $\langle \vec{\tau}, \vec{a} \rangle \equiv -1$.

We now assume that $\Sigma$ is marginally trapped and that its mean curvature mean curvature is future pointing, i.e.

$$\vec{H} = \langle \vec{H}, \vec{b} \rangle \vec{a}. \tag{17}$$

If the surface $Y$ is subjected to a normal variation, then the first variation formula gives

$$\begin{aligned}\partial_t \mathcal{A}[Y(t)] &= -2\int_\Sigma \langle \vec{H}, \partial_t Y\rangle \, d\Sigma \\ &= -2\int_\Sigma \langle \vec{H}, \vec{b}\rangle\langle \vec{a}, \partial_t Y\rangle \, d\Sigma \,.\end{aligned}$$

Requiring the flow to decrease area would suggest defining a flow by

$$\partial_t Y = \langle \vec{H}, \vec{b}\rangle \vec{b} + \alpha \vec{a}\,, \tag{18}$$

where the function $\alpha$ is chosen so that the flow preserves the property that the surface is marginally trapped. If this is possible, the integrand in the last integral above is $\langle \vec{H}, \vec{b}\rangle^2$ and the area will be decreasing under the flow. We will next show that under certain restrictions, it is possible to find $\alpha$ with the desired property.

For a variation of the surface generated by a section $S = \alpha\vec{a} + \beta\vec{b} \in \perp Y$, the first order change in the mean curvature is given by

$$\delta\vec{H} = J[S]\,,$$

where $J$ denotes the Jacobi operator of the surface. In [1] it was shown that, in terms of the null frame introduced above, the Jacobi operator can be expressed as

$$\begin{aligned}J[S] &= \big((L+A)[\alpha] + \langle C\vec{b}, \vec{b}\rangle\beta\big)\vec{a} \\ &\quad + \big((L-A)[\beta] + \langle C\vec{a}, \vec{a}\rangle\alpha\big)\vec{b}\,.\end{aligned} \tag{19}$$

Here, $L$ is a second order, elliptic, self-adjoint operator given by

$$L[u] = \Delta u + u\big[\frac{1}{2}(\langle \Delta^\perp \vec{a}, \vec{b}\rangle + \langle \Delta^\perp \vec{b}, \vec{a}\rangle) + \langle (B - \mathrm{Ric}^\perp)\vec{a}, \vec{b}\rangle\big],$$

where $\Delta^\perp$ and $\mathrm{Ric}^\perp$ are, respectively, the Laplacian and the Ricci tensor in the normal bundle and $B$ is the tensor defined by $\langle B(\xi), \eta\rangle := \langle \nabla^T \xi, \nabla^T \eta\rangle$ for $\xi$, $\eta \in \perp Y$. Also, $A$ is the first order skew-adjoint (with respect to the $L^2$ metic) operator, given by

$$A[u] := \frac{u}{2}\big[\frac{1}{2}(\langle \Delta^\perp \vec{a}, \vec{b}\rangle - \langle \Delta^\perp \vec{b}, \vec{a}\rangle) + 2\langle \nabla^\perp_{\nabla u} \vec{a}, \vec{b}\rangle\big]\,.$$

The tensor $C$ is given by

$$\langle C\xi, \eta\rangle := \langle B\xi, \eta\rangle + \mathrm{Ric}_{\mathcal{L}}(\xi, \eta)$$

As explained in [1], the Null Convergence Condition implies that $\langle C\eta, \eta\rangle \geqslant 0$ holds for all null normal sections $\eta$. We will now make the assumption that $\langle C\vec{a}, \vec{a}\rangle > 0$ holds. In the case that $\mathcal{L}$ is the Lorentz-Minkowsi space, this means that $\langle B\vec{a}, \vec{a}\rangle > 0$ holds, which means that the curvature of $\Sigma$ is non-vanishing. This turns out to be consistent with the surface locally being a spherical graph.

The flow (18) will preserve the equation (17) if and only if

$$\langle \vec{a}, \partial_t \vec{H}\rangle + \langle \partial_t \vec{a}, \vec{H}\rangle \equiv 0\,,$$

holds. (Here, the differentiation $\partial_t$ is used as a shorthand for covariant differentiation with respect to $\partial_t Y$ of the field $\vec{a}$, which is obtained by extending $a$ to be a field defined on the hypersuface defined by the flow (18).) However, we have from

$\langle \vec{a}, \vec{a} \rangle \equiv 0$ that $\langle \partial_t \vec{a}, \vec{a} \rangle \equiv 0$, so the second term is zero by (17). From (19) and (18), the flow will preserve (17) provided

$$(L - A)[\langle \vec{H}, \vec{b} \rangle] + \langle C\vec{a}, \vec{a} \rangle \alpha = 0$$

holds. Since we have assumed $\langle C\vec{a}, \vec{a} \rangle > 0$, we can solve the previous equation for $\alpha$ and we obtain a flow with all the desired properties:

$$\partial_t Y = -\frac{(L - A)[\langle \vec{H}, \vec{b} \rangle]}{\langle C\vec{a}, \vec{a} \rangle} \vec{a} + \langle \vec{H}, \vec{b} \rangle \vec{b}\ .$$

It is not difficult to see that, in the case where $\mathcal{L}$ is the Lorentz-Minkowski space, this flow agrees with the one considered in the first part of this paper. At the present time, we cannot assert even a short time existence result for this flow.

## 4. Discussion and Conclusions

Above, we have produced a convergent flow for marginally trapped surfaces in $\mathbf{R}^4_1$ under which they evolve to zero mean curvature surfaces. The flow is evidently not mean curvature flow since its principal part is of order four. We now give a geometric interpretation of the flow.

Consider a surface $X : S \to \mathbf{R}^3$ with non vanishing curvature. We regard S as a source for light rays traveling in a homogeneous, isotropic medium. By Huygen's Principle, the light ray through $X \in S$ can be identified with the null ray $t \mapsto (X, 0) + t(\nu, 1)$ in $\mathbf{R}^4_1$, where $\nu$ is the surface normal at X. Thus, the totality of light rays emitted by the surface can be regarded as a real line bundle whose fibers are null lines in $\mathbf{R}^4_1$. We look for a section of this bundle with a particular property, specifically, the section, regarded as a surface in $\mathbf{R}^4_1$, should be marginally trapped. In fact, if $q := X \cdot \nu$ denotes the support function of S, then, since

$$X = Dq + q\nu\ , \tag{20}$$

this marginally trapped section is given exactly by Y in formula (2).

We now define the functional $\mathcal{A}$ which assigns to the surface X the area of the surface Y. Equation (4) is evidently the Euler-Lagrange equation and it is well known that, with respect to the surface S, (4) is expressed as $\Delta(k_1^{-1} + k_2^{-1}) = 0$, where $k_i$ are the principal curvatures and $\Delta$ is the Laplacian with respect to the third fundamental form. For the functional $\mathcal{A}$, and the flow given by (9) induces a flow of the surface X which, by (20, satisfies

$$(\partial_t X)^{\perp} = -\frac{1}{2}(\frac{1}{k_1} + \frac{1}{k_2})\ \nu\ ,$$

i.e. it is the gradient flow for the functional A. Of course, as X evolves, Y evolves also.

## References


[1] Alías, Luis J., and Bennett Palmer. "Deformations of stationary surfaces." Classical and Quantum Gravity 14, no. 8 (1997): 2107.

[2] Aubin, Thierry. Nonlinear analysis on manifolds: Monge-Ampere equations. Vol. 252. New York: Springer-Verlag, 1982.

[3] Blaschke, Wilhelm. ”Vorlesungen über Differential Geometrie. Vol. III.” (1929).
[4] Guilfoyle, Brendan, and Wilhelm Klingenberg. ”A Converging Lagrangian Curvature Flow in the Space of Oriented Lines.” arXiv preprint arXiv:1310.4402 (2013).
[5] Palmer, Bennett. ”Area minimization among marginally trapped surfaces in Lorentz - Minkowski space.” Calculus of Variations and Partial Differential Equations 41, no. 3-4 (2011): 387-395.
[6] Penrose, Roger. Techniques of differential topology in relativity. Vol. 7. Philadelphia, Pennsylvania: Society for Industrial and Applied Mathematics, 1972.
[7] Penrose, Roger. ”Gravitational collapse and space-time singularities.” Physical Review Letters 14, no. 3 (1965): 57-59.

Bennett PALMER
Department of Mathematics, Idaho State University, Pocatello, ID 83209, U.S.A.
E-mail: palmbenn@isu.edu