STABILITY OF SPHERICALLY CONFINED FREE BOUNDARY DROPS WITH LINE TENSION

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ABSTRACT. We study the geometry of a stable drop of incompressible liquid constrained to lie in a spherical container. The energy functional is comprised of the surface tension, the wetting energy and the line tension. It is shown that the only stable equilibrium drops having the topology of a disc are flat discs and spherical caps. Sharp conditions for the stability of equilibrium spherical caps and flat discs are given.

Keywords: Stability, wetting, line tension, free boundary, Wentzell boundary condition
AMS classification: 53A10, 35J20

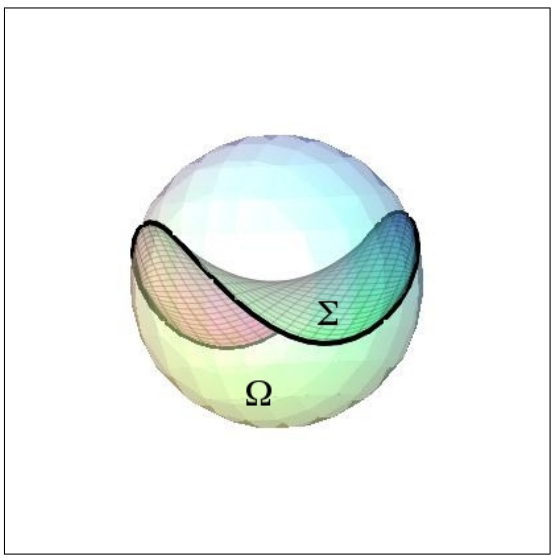


FIGURE 1. The basic three phase configuration.

1. INTRODUCTION

The determination of the equilibrium surface having free boundary on a supporting surface is a subject with a long and interesting history which probably originates with Da Vinci's investigation of the surface of a capillary tube. The basic three phase system consists of a partially filled container, in the present case, a spherical container partially filled by a liquid. Josiah Gibbs observed that points on the free boundary curve which is given by the interface of the liquid's surface with the container, being in contact with three distinct phases, should carry an energy term of their own; the so called line tension. For isotropic materials, this energy is often represented by a multiple of the length of the boundary curve, although other possibilities can be used. Since this line tension scales linearly while the surface energy scales quadratically, the line tension becomes increasingly more significant at small scales. In addition, at small scales the gravitational force, which scales cubically, is negligible so it is omitted here.

In this paper, we study the stability of equilibria for a liquid drop contained within a spherical container as in Figure 1. Since the drop is liquid, the energy of its free surface Σ is proportional to the area and we normalize the constant of proportionality to be one. The interaction of the material of the drop with that of the container contributes a wetting energy which is proportional to the area of the spherical region Ω in contact with the bulk of the liquid. Points on the free boundary curve $\partial\Sigma$ are in contact with three phases; the drop, the container and the material, assumed to be air, which occupies the region interior to the sphere which is complimentary to the drop. These points contribute an energy term, the line tension discussed above, which we assume to be proportional to the length of $\partial\Sigma$. We thus arrive at the total energy

$$E := \mathrm{Area}[\Sigma] + \omega \mathrm{Area}[\Omega] + \beta \mathrm{Length}[\partial\Sigma]\,, \tag{1}$$

where ω and β are coupling constants. The main result presented here is the following:

Theorem 1.1. *Let $X : (\Sigma, \partial\Sigma) \to (B^3, S^2)$ be a C^2 immersed equilibrium drop where Σ is the unit disc in $\mathbf{R}^2$. Then, if the surface is stable, $X(\Sigma)$ is a spherical cap or a flat disc.*

Nitsche [10] considered the free boundary problem without wetting or line tension, ($\omega = \beta = 0$), and gave a beautiful complex analytic argument to show that the only equilibrium disc type solutions, stable or otherwise, are spherical caps and flat discs. In [11], Ros and Souam extended this result to drops with wetting energy but no line tension ($\beta = 0$).

For genus zero surfaces contained in the upper half space $x_3 \geqslant 0$ and having one free boundary component in the plane $x_3 = 0$, it is known that for $\beta \geqslant 0$, all equilibrium surfaces are spherical caps [7]. When $\beta < 0$ holds, it was shown by B. Widom [15], that the equilibrium spherical caps are never stable. These surfaces are however energy minimizing among rotationally symmetric surfaces. It was pointed out in [13] that this mathematical instability does not preclude the physical existence of drops with negative line tension, since the wave lengths of destabilizing variations may lie below the length scale for which this type of energy model is valid.

In the second part of the paper, we study the stability of spherical caps. We give a necessary and sufficient condition for a spherical cap to be a stable equilibrium for a functional of the type (1).

2. Stability of disc type equilibria

In order to obtain the equations characterizing equilibrium, we subject the surface Σ to a variation $\mathbf{X}_\epsilon = \mathbf{X} + \epsilon\delta\mathbf{X} + \mathcal{O}(\epsilon^2)$. Letting $\mathbf{X}$ denote the position vector of the surface, we write $\delta\mathbf{X} = \mathbf{T} + \psi\mathbf{N}$, where $\mathbf{N}$ denotes the surface normal and $\mathbf{T}$ is tangent to Σ. The field δX must, in addition, be tangent to the sphere S^2 along $\partial\Sigma$, i.e.

$$\mathbf{X}\cdot\delta\mathbf{X} \equiv 0\,, \text{ on } \partial\Sigma \tag{2}$$

and it must satisfy the condition

$$\int_\Sigma \psi \; d\Sigma = 0 \tag{3}$$

which means that the variation infinitesimally preserves the volume of the drop. A standard argument using the Implicit Function Theorem [2] can be applied to show that conditions (2) and (3) are sufficient to produce an actual variation $\mathbf{X} + \epsilon\delta\mathbf{X} + \mathcal{O}(\epsilon^2)$ which preserves the volume and keeps the drop confined to the sphere.

The first variation of the total energy is

$$\delta\text{Area}[\Sigma] = -\int_\Sigma 2H\psi \; d\Sigma + \oint_{\partial\Sigma} \delta\mathbf{X}\cdot\mathbf{n}\; ds\,, \tag{4}$$

where $\mathbf{n} = \mathbf{X}' \times \mathbf{N}$ is the unit conormal to $\partial\Sigma$ which it tangent to Σ. ("prime" denotes differentiation with respect to arc length along $\partial\Sigma$) and H is the (scalar) mean curvature. The variation of the wetting energy is [11]

$$\delta\text{Area}[\Omega] = \oint_{\partial\Sigma} \delta\mathbf{X}\cdot\bar{\mathbf{n}}\; ds\,, \tag{5}$$

where $\bar{\mathbf{n}} = \mathbf{X}\times\mathbf{X}'$ is the unit conormal to $\partial\Sigma$ in S^2. We point out that our orientation of the surface Ω is different from that of the surface Σ since $\mathbf{n}\times\mathbf{X}' = \mathbf{N}$, while $\bar{\mathbf{n}}\times\mathbf{X}' = -\mathbf{X}$. Along $\partial\Sigma$, there holds $\mathbf{X}'' = -\mathbf{X} + \bar{k}_g\bar{\mathbf{n}}$, where $\bar{k}_g$ is the geodesic curvature of $\partial\Sigma$ in S^2. The variation of the line tension is

$$\delta\text{Length}[\partial\Sigma] = -\oint_{\partial\Sigma} \mathbf{X}''\cdot\delta\mathbf{X}\; ds\,. \tag{6}$$

By considering variations with compact support satisfying (3), we can conclude that $H \equiv$ constant in the interior of Σ. Then, collecting the boundary terms, we conclude that

$$0 = \oint_{\partial\Sigma} (\mathbf{n} + \omega\bar{\mathbf{n}} + \beta(\mathbf{X} - \bar{k}_g\bar{\mathbf{n}}))\cdot\delta\mathbf{X}\; ds$$

must hold for all admissible $\delta\mathbf{X}$. This will hold if and only if $\mathbf{n} + (\omega - \beta\bar{k}_g)\bar{\mathbf{n}} + \beta\mathbf{X}$ is parallel to $\mathbf{X}$ along $\partial\Sigma$, i.e.

$$\mathbf{X}\times(\mathbf{n} + (\omega - \beta\bar{k}_g)\bar{\mathbf{n}}) = 0 \tag{7}$$

must hold. Since $\mathbf{X}\times\bar{\mathbf{n}} = -\mathbf{X}'$ and $\mathbf{X}\times\mathbf{n} = (\mathbf{X}\cdot\mathbf{N})\mathbf{N}\times\mathbf{n} = (\mathbf{X}\cdot\mathbf{N})\mathbf{X}'$, we see that (7) is equivalent to

$$\mathbf{X}\cdot\mathbf{N} = -\beta\bar{k}_g + \omega\,, \tag{8}$$

and the first variation is given by

(9) $$\delta E = -\int_\Sigma 2H\psi \, d\Sigma + \oint_{\partial\Sigma} \big[\mathbf{X}\cdot\mathbf{N} + \beta\bar{k}_g - \omega\big]\mathbf{X}' \times \mathbf{X}\cdot\delta\mathbf{X} \, ds \, .$$

Obviously, axially symmetric CMC surfaces contained in the ball having their boundary circles on the sphere provide examples of equilibrium surfaces, Partial examples, i.e. constant mean curvature surfaces having a boundary arc on the sphere where (8) holds, can be produced using Bjorling's Formula [6]. This formula gives the construction of a minimal surface containing a given 'strip' which consists of a real analytic curve $C(s)$, parameterized by arc length, together with a unit vector field $\eta(s)$ orthogonal to the curve at each point. If $\mathbf{X}(w)$ denotes the minimal immersion and defined on a neighborhood of the real axis in the complex plane, then the formula is

(10) $$X(w) := \Re\{C(w) - i\int_{w_0}^{w} \eta(z) \times C'(z) \, dz\} \, .$$

Here $C(w)$ and $\eta(w)$ denote the analytic continuations of $C(s)$ and $\eta(s)$ respectively.

We take $\mathbf{C} : I \to S^2$ be curve which is real analytic in its arc length parametrization such that its geodesic curvature satisfies $|-\beta\bar{k}_g + \omega| < 1$ for constants ω, β. We also use $\bar{k}_g$ to denote the analytic extension of the geodesic curvature to a neighborhood of I in the complex plane. We then define

$$\eta(s) = -(\omega - \beta\bar{k}_g)\mathbf{C}(s) - \sqrt{1-(\beta\bar{k}_g-\omega)^2}\, \mathbf{C}\times\mathbf{C}'(s) \, .$$

The field η will turn out to be minus the unit normal $\mathbf{N}$ restricted of the minimal surface restricted to the curve $C(s)$.

Plugging the field η into (10), we get that

$$\mathbf{X}(w) := \mathrm{Re}\{\mathbf{C}(w) - i\int_{w_0}^{w} (\omega-\beta\bar{k}_g)\mathbf{C}(w)\times\mathbf{C}'(s) + \sqrt{1-(\beta\bar{k}_g-\omega)^2}\,\mathbf{C}(z)\, dz\}$$

defines a minimal surface satisfying (8) on $\mathbf{C}(I)$. Locally, the curve $\mathbf{C}(I)$ divides the minimal surface into two parts, on interior to the 2-sphere and one exterior. The interior part is then a minimal surface having a boundary component satifying (8). This construction is useful since it shows that there is no local obstruction to having an equilibrium surface with a non circular boundary arc.

The second variation of the functional E was worked out in detail in the paper [12], however we do the calculations for the special case where the supporting surface is a sphere in the appendix. Assuming the surface Σ to be in equilibrium, the second variation of energy is given by.

(11) $$\delta^2 E = -\int_\Sigma \psi L[\psi] \, d\Sigma + \oint_{\partial\Sigma} \psi B[\psi] \, ds \, .$$

Here $L = \Delta + (4H^2 - 2K)$, where K is the Gaussian curvature of the surface and Δ denotes the Laplace-Beltrami operator. With $\sin\alpha := \mathbf{X}\cdot\mathbf{n}$, the boundary operator has the form

(12)

$$B[\psi] = \nabla\psi\cdot\mathbf{n} - \big(\frac{1}{\sin\alpha} - \cot\alpha\, d\mathbf{N}(\mathbf{n})\cdot\mathbf{n}\big)\psi - \frac{\beta}{\sin\alpha}\big((\frac{\psi}{\sin\alpha})_{ss} + (1+\bar{k}_g^{\,2})\frac{\psi}{\sin\alpha}\big).$$

At points on the boundary where $\mathbf{X}\cdot\mathbf{n} =: \sin\alpha = 0$, $\psi/\sin\alpha$ must be replaced by $-\mathbf{T}\cdot\mathbf{n}$, where $\mathbf{T}$ is the tangential part of $\delta\mathbf{X}$. Defining $\tilde{\psi} := \psi|_{\partial\Sigma}/\sin\alpha$ and integrating by parts, the second variation can now be expressed

(13)

$$\delta^2 E = \int_\Sigma |\nabla\psi|^2 - |d\mathbf{N}|^2\psi\, d\Sigma + \oint_{\partial\Sigma} \beta[(\tilde{\psi}_s)^2 - \tilde{\psi}^2] - \psi^2[\csc\alpha - \cot(\alpha)II(\mathbf{n},\mathbf{n})]\, ds$$

This formula extends (11) to the class of functions

$$\mathcal{K} := \{\psi \in H^1(\Sigma) \,|\, \tilde{\psi} := \psi/\sin\alpha \in H^1(\partial\Sigma)\}\,.$$

Definition An equilibrium surface is *stable* if $\delta^2 E \geqslant 0$ holds for all $\psi \in \mathcal{K}$ such that (3) holds.

Diagonalizing the second variation leads to a spectral problem of the form

$$(L+\lambda)\psi = 0, \text{ on } \Sigma\,, B[\psi] = 0 \text{ on } \partial\Sigma\,.$$

The second condition is a type of Wentzell boundary condition, has been widely studied [3], [8], [14].

It will be assumed that Σ is an embedded topological disc which is in equilibrium for the functional E such that $\partial\Sigma$ is a piecewise smooth curve. Our aim is to show that if Σ is stable, then Σ must be axially symmetric. For $\mathbf{c} \in (\mathbf{R}^3)^*$, the field $\mathbf{c}\times\mathbf{X}$ generates an infinitesimal rotation with axis $\mathbf{c}$. It is clear that this flow, being a one parameter family of isometries, preserves volumes, areas and arc lengths. Introduce the function $\psi_{\mathbf{c}} := \mathbf{c}\times\mathbf{X}\cdot\mathbf{N}$. It is then clear that $\psi_{\mathbf{c}}$ satisfies $L[\psi_{\mathbf{c}}] = 0$ in Σ with $B[\psi_{\mathbf{c}}] = 0$ on $\partial\Sigma$. Also note that (3) holds since rotations preserve volume and (2) is obviously satisfied. We will need the following:

Lemma 2.1. *If there exists an arc γ in the boundary of the disc on which $\psi_{\mathbf{c}} \equiv 0$ holds, then the surface is axially symmetric and it must be a flat disc or a spherical cap.*

Proof. Assuming that the hypotheses of the lemma hold, at least one of the following two statements must be true: (i) there is an open arc $\gamma_1 \subset \gamma$ on which $\mathbf{X}\cdot\mathbf{n}$ is nowhere zero. (ii) there is an open arc $\gamma_1 \subset \gamma$ on which $\mathbf{X}\cdot\mathbf{n} \equiv 0$ holds.

In case (i), write

$$0 \equiv \psi_{\mathbf{c}} = \mathbf{c}\times\mathbf{X}\cdot\mathbf{N} = \mathbf{c}\times\mathbf{X}\cdot\mathbf{n}\times\mathbf{X}' = -(\mathbf{X}\cdot\mathbf{n})(\mathbf{c}\cdot\mathbf{X}')\,, \tag{14}$$

so $\mathbf{c}\cdot\mathbf{X}' \equiv 0$ holds on γ_1. It follows that $\mathbf{X}(\gamma_1)$ lies in the intersection of S^2 with a plane, so it is a circular arc.

Since $\mathbf{X}(\gamma_1)$ is a circular arc, $\bar{k}_g$ is constant and we get from (8)

$$0 = \partial_s(\mathbf{X}\cdot\mathbf{N} + \beta\bar{k}_g - \omega) = \partial_s(\mathbf{X}\cdot\mathbf{N}) = \mathbf{X}\cdot(-\tau_g\mathbf{n} - k_N\mathbf{X}') = -\tau_g\mathbf{X}\cdot\mathbf{n}\,, \tag{15}$$

where τ_g denotes the geodesic torsion of $\partial\Sigma$ in Σ and k_N denotes the normal curvature. This means that $\tau_g \equiv 0$ holds on γ_1. We use this to compute the normal derivative of $\psi_{\mathbf{c}}$ on γ_1:

$$\begin{aligned}
\partial_{\mathbf{n}}\psi_{\mathbf{c}} &= \mathbf{c}\times\mathbf{n}\cdot\mathbf{N}+\mathbf{c}\times\mathbf{X}\cdot d\mathbf{N}(\mathbf{n})\\
&= -\mathbf{N}\times\mathbf{n}\cdot\mathbf{c}-(2H-k_N)\mathbf{c}\times\mathbf{X}\cdot\mathbf{n}-\tau_g\mathbf{c}\times\mathbf{X}\cdot\mathbf{X}'\\
&= -\mathbf{X}'\cdot\mathbf{c}-(2H-k_N)\mathbf{X}'\cdot\mathbf{c}-\tau_g\mathbf{c}\times\mathbf{X}\cdot\mathbf{X}'\\
&= 0\,.
\end{aligned}$$

Now we have that $\psi_{\mathbf{c}}\equiv 0\equiv\partial_{\mathbf{n}}\psi_{\mathbf{c}}$ along γ_1 which is a sub arc of the unit circle. We can then use the Cauchy- Kovaleskaya Theorem to conclude that $\psi_{\mathbf{c}}\equiv 0$ holds on Σ. Recall that by elliptic regularity, CMC surfaces in $\mathbf{R}^3$ are real analytic ([9], Theorem 6.6.1) and hence the coefficients of the operator L are real analytic with respect to the standard coordinates on the disc. The CMC surface Σ can be analytically continued across the arc γ_1. The technique to do this, given in [5] for minimal surfaces, extends to the CMC case. The Cauchy problem $L[f]=0$ with $f\equiv 0\equiv\partial_n f$ on γ_1 is locally well posed and we can conclude that $f\equiv 0$ is the unique solution in a neighborhood of any point in γ_1. Therefore $\psi_{\mathbf{c}}\equiv 0$ in a neighborhood of any point in γ_1 and so by uniqueness of analytic continuation $\psi_{\mathbf{c}}\equiv 0$ holds in Σ.

Since $\psi_{\mathbf{c}}\equiv 0$ holds, it follows that Σ is axially symmetric about the vector $\mathbf{c}$, since the torque field $\mathbf{X}\times\mathbf{c}$ is everywhere tangent to the surface, so each of its integral curves, which are coaxial circles perpendicular to $\mathbf{c}$, are contained in the surface. Since the surface has constant mean curvature and is assumed to be a topological disc, Σ must be either a spherical cap or a flat disc.

If case (ii) holds, we must have $\mathbf{X}\cdot\mathbf{N}\equiv\pm 1$ on γ_1, since $\mathbf{X}\cdot\mathbf{X}'\equiv 0$ holds. Consquently $\mathbf{N}'=\pm\mathbf{X}'$ implies that $k_N\equiv\pm 1$ and $\tau_g\equiv 0$ holds. We again get from (8) that $\bar{k}_g\equiv$ constant on $\mathbf{X}(\gamma_1)$. Since the curvature κ of $\mathbf{X}(\gamma_1)$ as a space curve satisfies $\kappa^2=1+\bar{k}_g^2$, $\kappa\equiv$ constant holds. We can then easily obtain from $\mathbf{X}''=-\mathbf{X}+\bar{k}_g\bar{\mathbf{n}}$, that $\mathbf{X}'''$ and $\mathbf{X}'$ are parallel. Since the torsion of $\mathbf{X}(\gamma_1)$ is given $\mathbf{X}'\times\mathbf{X}''\cdot\mathbf{X}'''/|\mathbf{X}'\times\mathbf{X}''|$, the torsion is zero and so $\mathbf{X}(\gamma_1)$ is a circular arc.

Note that when $\mathbf{X}\cdot\mathbf{n}\equiv 0$ holds, a calculation similar to that done in (14) shows that $\psi_{\mathbf{a}}:=\mathbf{a}\times\mathbf{X}\cdot\mathbf{N}$ vanishes identically along γ_1 for *any* non zero vector $\mathbf{a}$. We take $\mathbf{a}$ to be a non zero vector which is perpendicular to the plane containing γ_1. The same steps in the calculation for $\partial_{\mathbf{n}}\psi_{\mathbf{c}}$ done above then show that $\partial_{\mathbf{n}}\psi_{\mathbf{a}}\equiv 0$ holds along γ_1 and by the same reasoning, the surface is axially symmetric and must be a spherical cap or flat disc. **q.e.d**

Proof of Theorem 1.1 On $\partial\Sigma$, there holds $\psi_{\mathbf{c}}=\mathbf{c}\times\mathbf{X}\cdot\mathbf{N}=-\mathbf{N}\times\mathbf{X}\cdot\mathbf{c}=-(\mathbf{X}\cdot\mathbf{n})\mathbf{N}\times\mathbf{n}\cdot\mathbf{c}=-(\mathbf{X}\cdot\mathbf{n})\mathbf{X}'\cdot\mathbf{c}$. We will use this to show that if we assume the surface is not axially symmetric, then there always exists a $\mathbf{c}\in(\mathbf{R}^3)^*$ such that the function $\psi_{\mathbf{c}}$ has, at least, four sign changes on $\partial\Sigma$.

If (r,θ) are the polar coordinates on the disc, then $\mathbf{X}'=\mathbf{X}_\theta/|\mathbf{X}_\theta|$ and so

$$\psi_{\mathbf{c}}|_{\partial\Sigma}=-(\mathbf{X}\cdot\mathbf{n}/|\mathbf{X}_\theta|)\mathbf{X}_\theta\cdot\mathbf{c}\,. \tag{16}$$

For any $\mathbf{c}$

$$\oint_{\partial\Sigma} \mathbf{X}_\theta \cdot \mathbf{c}\, d\theta = \oint_{\partial\Sigma} (\mathbf{X}\cdot\mathbf{c})_\theta\, d\theta = 0$$

holds. Let

$$\oint_{\partial\Sigma} \mathbf{X}_\theta e^{i\theta}\, d\theta =: \mathbf{A} + i\mathbf{B} \in \mathbf{C}^3 .$$

Thus, there exists $\mathbf{c} \in \mathbf{R}^3$, $\mathbf{c} \neq \mathbf{0}$ with $0 = \mathbf{c}\cdot\mathbf{A} = \mathbf{c}\cdot\mathbf{B}$. It then follows that for this $\mathbf{c}$ the function $\mathbf{X}_\theta \cdot \mathbf{c}$ can be represented as a Fourier series of the form

$$\mathbf{X}_\theta \cdot \mathbf{c} = \sum_{j\geqslant 2} (a_j \cos(j\theta) + b_j \sin(j\theta)) .$$

Since the disc is simply connected, this function can be interpreted as the boundary values of the real part of the complex analytic function

$$F(z) := \sum_{j\geqslant 2} (a_j - ib_j) z^j .$$

Note that $F(z) = z^2\eta(z)$ for a function $\eta(z)$ which is complex analytic in the disc. The variation of $\arg(F(z))$ over the boundary of the disc is therefore at least 4π. If $\mathrm{Re}(F(z))$ vanishes identically on an open arc in $\partial\Sigma$, then the same is true for $\psi_{\mathbf{c}}$ and, by Lemma 2.1, the surface is axially symmetric so we can assume that this does not happen. Because the variation of $\arg(F(z))$ is at least 4π, $\mathrm{Re}(F(z))$ must have at least four sign changes on $\partial\Sigma$. Specifically, there are at least four points p_i which are contained in arcs on which $\mathrm{Re}(F(z))$ is positive on one side of p_i and negative on the other side. The same is true for the function $\psi_{\mathbf{c}}$. This follows from equation (16) and the fact that $\mathbf{X}\cdot\mathbf{n} \geqslant 0$ holds on $\partial\Sigma$ since the function $\|\mathbf{X}\|^2$ clearly assumes its maximum on $\partial\Sigma$. Also recall that $\mathbf{X}\cdot\mathbf{n}$ cannot vanish on any arc in $\partial\Sigma$ by Lemma 2.1 and (16).

At each point p_i, at least one arc of the nodal set of $\psi_{\mathbf{c}}$ must enter Σ. These arcs must divide the disc into at least three nodal domains of the function $\psi_{\mathbf{c}}$. We will now show that the existence of more than two nodal domains of $\psi_{\mathbf{c}}$ implies instability. Suppose $\Omega_1, \Omega_2, ...\Omega_N$ are nodal domains, $N \geqslant 3$. We can assume Ω_1 and Ω_2 are adjacent to each other. Let $U = \Omega_1 \cup \Omega_2$, $V = \Sigma \setminus U$.

Define $\mathcal{F}$ to be the set of all functions f on $\bar{U}$ satisfying the following conditions: (i) f is piecewise C^1 on $\bar{U}$, (ii) $f/\mathbf{X}\cdot\mathbf{n}$ is piecewise C^1 on $\partial\Sigma$ and (iii) $f \equiv 0$ on $\partial U \setminus \partial\Sigma$. Note that $\psi_c|_U \in \mathcal{F}$. Also, since $(\mathbf{X}\cdot\mathbf{N})^2 + (\mathbf{X}\cdot\mathbf{n})^2 \equiv 1$ on $\partial\Sigma$, $\mathcal{F}$ contains all functions vanishing identically on V which are of the form $\nu(1-(\mathbf{X}\cdot\mathbf{N})^2)$ near $\partial U \cap \partial\Sigma$, where ν is smooth function. In particular, this includes $C_c^\infty(U)$. Define

$$\begin{aligned}\mu_1 \quad &= \inf_{\mathcal{F}} \Bigg(\int_U |\nabla f|^2 - (4H^2 - 2K) f^2\, d\Sigma - \int_{\partial\Sigma\cap\partial U} \Big(\frac{1}{\sin\alpha} - \cot\alpha\, d\mathbf{N}(\mathbf{n})\cdot\mathbf{n} \Big) f^2\, ds \\ &\quad + \beta \int_{\partial\Sigma\cap\partial U} \hat{f}_s^2 - \hat{f}^2\, ds \Bigg) \Bigg/ \int_U f^2\, d\Sigma .\end{aligned}$$

By using the function $\psi_{\mathbf{c}}|_U$, we get that $\mu_1 \leqslant 0$ holds.

Suppose $\mu_1 = 0$. Let ψ^* be the function which is identically $\psi_{\mathbf{c}}$ in Ω_1 and is identically zero off Ω_1. Then ψ^* realizes the infimum $\mu_1 = 0$ and hence we have, for all $\zeta \in C_c^\infty(U)$,

$$\begin{aligned} 0 &= \partial_\epsilon \Bigg(\int_U |\nabla(\psi^* + \epsilon\zeta)|^2 - (4H^2 - 2K)(\psi^* + \epsilon\zeta)^2 \, d\Sigma \\ &\quad - \int_{\partial\Sigma\cap\partial U} \big(\frac{1}{\sin\alpha} - \cot\alpha \, d\mathbf{N}(\mathbf{n}) \cdot \mathbf{n}\big)(\psi^* + \epsilon\zeta)^2 \, ds \\ &\quad + \beta \int_{\partial\Sigma\cap\partial U} (\hat{\psi}^* + \epsilon\hat{\zeta})_s^2 - (\hat{\psi}^* + \epsilon\hat{\zeta})^2 \, ds \Bigg)_{\epsilon=0} \\ &= 2\int_U \nabla\psi^* \cdot \nabla\zeta - (4H^2 - 2K)\psi^*\zeta \, d\Sigma \, . \end{aligned}$$

In other words, ψ^* is a weak solution of $L = 0$ in U. Elliptic regularity then implies that ψ^* is a classical solution. However $\psi^* \equiv 0$ on Ω_2 which contradicts a well known unique continuation property [1].

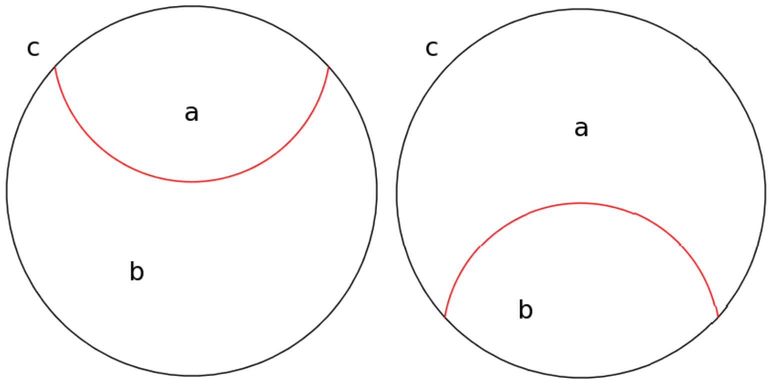
We can therefore conclude that μ_1 is negative and so there is then an $f \in \mathcal{F}$ for which the ratio in (17) is negative. Extend f to be zero in V. Let ψ_2 be the function which is identically equal to $\psi_{\mathbf{c}}$ in V and is identically zero in U. There is a nontrivial superposition $\phi := c_1 f + c_2\psi_2 \in \mathcal{K}$ which has zero mean value. We seek a variation field $\delta\mathbf{X} = \phi\mathbf{N} + \mathbf{T}$, with $\mathbf{T}$ tangent to Σ such that on $\partial\Sigma$ there holds $0 \equiv \delta\mathbf{X} \cdot \mathbf{X} = \phi\mathbf{N} \cdot \mathbf{X} + (\mathbf{T} \cdot \mathbf{n})(\mathbf{n} \cdot \mathbf{X})$. In other words, we want $\mathbf{T} \cdot \mathbf{n} = -(\phi/(\mathbf{n}\cdot\mathbf{X}))\mathbf{N}\cdot\mathbf{X}$. This last expression is well-defined and piecewise differentiable by the definition of $\mathcal{F}$. Let w be the solution of the biharmonic Dirichlet problem $\Delta^2 w = 0$ in Σ having boundary values $w \equiv 0$ and $\partial_{\mathbf{n}} w = -\phi/(\mathbf{n} \cdot \mathbf{X})$. Then setting $\delta\mathbf{X} := \nabla w + \phi\mathbf{N}$, we get $\delta\mathbf{X} \cdot \mathbf{X} \equiv 0$ on $\partial\Sigma$. Using a standard method ([2]), this variation field can be shown to arise from a genuine one parameter family of surfaces contained in the ball which enclose the same volume as Σ and for this variation, the second variation of energy would be negative. Thus, for the surface to be stable $\psi_{\mathbf{c}} \equiv 0$ must hold, in which case the surface is axially symmetric. **q.e.d.**

3. STABILITY OF EQULIBRIUM SPHERICAL CAPS

We consider a spherical container represented as the unit ball B. It is occupied two phases a and b, separated by a single interface Σ which is assumed to have the geometry of a spherical cap or flat disc. The exterior of B is occupied by a phase c. Each interface carries a surface energy which is proportional to its area. The coupling constants for the interface between phase x and phase y is denoted σ_{xy} with σ_{ab} normalized to be 1. The total energy is then given by

$$\begin{aligned} \mathcal{E} &= \text{Area}[\Sigma] + \sigma_{ac}\text{Area}[\Omega] + \sigma_{bc}(4\pi - \text{Area}[\Omega]) + \beta\text{Length}[\partial\Sigma] \\ &= \text{Area}[\Sigma] + (\sigma_{ac} - \sigma_{bc})\text{Area}[\Omega] + \beta\text{Length}[\partial\Sigma] + 4\pi\sigma_{bc} \, . \qquad (17) \end{aligned}$$

There are two possible configurations, pictured in Figure 2, which can occur: It is clear from (17) that every configuration of type II is variationally equivalent to one



(A) I (B) II.

FIGURE 2. Two configurations of spherical caps.

of type I, but with a different coefficient ω since the volume preserving variations of Σ must preserve the volumes of both the a and b phases. For this reason, *we will only consider the configurations of type I*. Equilibrium discs can be considered as surfaces of either type.

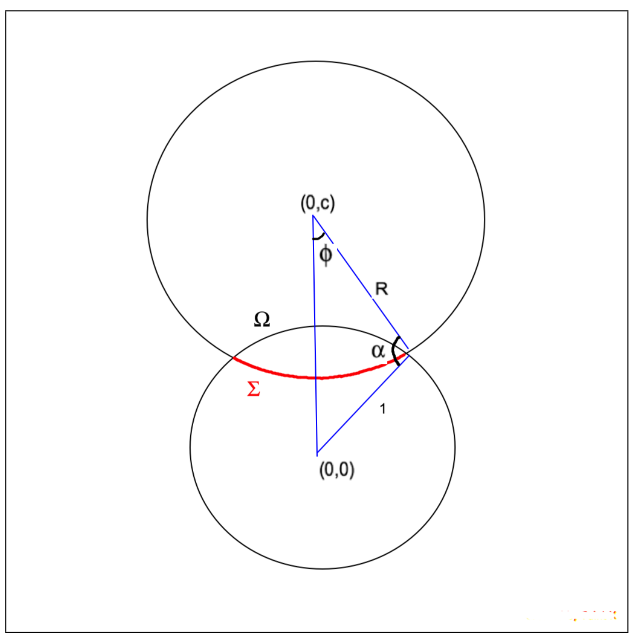


FIGURE 3. Spherical cap

First assume that the surface is not a flat disc. We represent a spherical cap Σ as the intersection of a sphere of radius R and center $(0, 0, c)$ with the unit ball centered at the origin. The energy is

$$E = 2\pi R^2(1-\cos\phi) + 2\pi\omega(1-\sqrt{1-R^2\sin^2\phi}) + 2\pi\beta R\sin\phi\,. \tag{18}$$

The boundary condition for equilibrium becomes

$$R(1-c\cos\phi) = \beta\frac{c-R\cos\phi}{\sqrt{1-[c-R\cos\phi]^2}} + \omega\,, \tag{19}$$

where c, R, ϕ are related by

$$1 = c^2 + R^2 - 2cR\cos\phi\,.$$

From Figure 3, we get $\cos\alpha + c\cos\phi = R$. Using this we get, after some manipulation, that the second variation formula for a spherical cap Σ is given by

$$\delta^2 E = -\int_\Sigma \psi(\Delta+\frac{2}{R^2})\psi\, d\Sigma + \oint_{\partial\Sigma}\psi\big(\psi_n - \frac{\cot\phi}{R}\psi\big)\, ds - \beta\oint_{\partial\Sigma}\hat\psi(\hat\psi'' + \frac{1}{R^2\sin^2\phi}\hat\psi)\, ds. \tag{20}$$

It is clear that a spherical cap which is in equilibrium for a single energy functional is, in fact, in equilibrium for a linear continuum of functionals obtained by varying (ω, β) so that the right hand side of (19) is left unchanged. A particular spherical cap will be stable for only a certain range of the parameters in this family. For example, all spherical caps are stable when $\beta = 0$, [11], but for $\beta < 0$, all spherical caps are unstable. As in the case of planar boundary, studied in [15], the surface can be deformed near the boundary so that the length increases while fixing the wetted area Ω. This type of deformation can be localized in a small neighborhood of the boundary curve so that it has negligible effect on the surface area.

It is also clear from (20), that for a fixed spherical cap, increasing β, (while decreasing ω so that the right hand side of (8) remains constant), will lead to instability, since ω does not appear in the second variation. The type of instability that occurs is referred to in [15] as a *drying transition* since as β increases, the drop will tend to detach from the supporting sphere in order to decrease energy. This will even happen for $\omega < 0$, since the circumference of the wetted disc dominates the wetted area as its radius tends to 0.

Lemma 3.1. *An equilibrium spherical cap for a functional as given in (1) with* $\beta \geqslant 0$ *is stable if and only if it is stable with respect to axially symmetric volume preserving variations.*

Proof. The necessity is clear. Consider the variational problem for a pendent drop having free boundary in a horizontal plane. If the total energy is

$$E = \text{Area}(\Sigma) + \hat\omega\text{Area}(W) + \hat\beta\text{Length}(\partial\Sigma) \tag{21}$$

where W is the wetted area in the plane, then the boundary equation for equilibrium is

$$\mathbf{N}\cdot\mathbf{E}_3 = \hat\omega + \frac{\hat\beta}{R\sin\phi}\,. \tag{22}$$

If we use $\mathbf{N} \cdot \mathbf{E}_3 = -\cos\phi$ and set $\hat{\beta} = \beta/\sin^2\alpha$ and then define $\hat{\omega}$ by (22), we find that an equilibrium spherical cap is also an equilibrium surface for the problem (21), (see [7]). In addition, the surface is simultaneously stable or unstable for both problems. However, it follows from Corollary 5.1 of [7] that an axially symmetric surface is stable for the functional (21) if and only if it is stable with respect to axially symmetric volume preserving variations. **q.e.d.**

Theorem 3.2. *A non flat equilibrium spherical drop of type I is stable if and only if*

$$0 \leqslant \beta \leqslant \beta_{\max} := c^2 R \sin^3\phi(2+\cos\phi) \,. \tag{23}$$

holds. Here c, ϕ *and* R *are as in Figure 3.*

An equilibrium disc of radius R *is stable if and only if*

$$0 \leqslant \beta \leqslant \beta_{\max} := 3R^3 \,. \tag{24}$$

Proof. By the lemma, we need only consider axially symmetric variations.

We will first consider the case where Σ is not a flat disc. Let s denote arc length from the pole of Σ and define

$$\psi_1 := N_3 - \frac{1}{\text{Area}(\Sigma)}\int_\Sigma N_3 \, d\Sigma = -\cos(s/R) - \frac{1}{2}(1+\cos\phi) \,. \tag{25}$$

This function is, up to a multiplicative constant, the unique axially solution of $\Delta u + 2R^{-2}u \equiv$constant having mean value zero on Σ. A calculation gives

$$\frac{\partial_{\mathbf{n}}\psi_1}{\psi_1} = \frac{\partial_s\psi_1}{\psi_1} = \frac{2\sin\phi}{R(1-\cos\phi)} \,,$$

on $\partial\Sigma$. Solving the equation $\frac{\cot\phi}{R} + \frac{\beta_{\max}}{R^2}\csc^2\alpha\csc^2\phi = \frac{2\sin\phi}{R(1-\cos\phi)}$ for $\beta_{\max}$ then gives that the second variation formula (20) vanishes for $\beta = \beta_{\max}$ and this particular choice of ψ. Clearly, for any $\beta > \beta_{\max}$, the same function will make the second variation negative since $\psi_1(R\phi) \neq 0$. This shows the necessity of the condition in the theorem.

To show the sufficiency, we will show that, when $\beta = \beta_{\max}$ the second variation $\delta_u^2 E$ attains its minimum value (zero) at $u = \psi$. Hence for $\beta \leqslant \beta_{\max}$, the second variation is non negative.

For axially symmetric functions and $\beta = \beta_{\max}$, the second variation can be expressed.

$$\begin{aligned} \delta_u^2 E &= \int_\Sigma \|\nabla u\|^2 - \frac{2}{R^2}u^2 \, d\Sigma \\ &\quad -2\pi R(\sin\phi)(u(R\psi))^2\Big(\frac{\cot\phi}{R} + \frac{\beta_{\max}}{R^2}\csc^2\phi\csc^2\alpha\Big) \\ &= \int_\Sigma \|\nabla u\|^2 - \frac{2}{R^2}u^2 \, d\Sigma \\ &\quad -2\pi R(\sin\phi)(u(R\psi))^2\Big(\frac{2\sin\phi}{R(1-\cos\phi)}\Big) \,. \end{aligned} \tag{26}$$

Note the homogeneity in u which allows us to normalize the functions.

Let $\mathcal{G}$ denote the class of smooth axially symmetric functions $u(s)$, $0 \leqslant s \leqslant R\phi$ on Σ satisfying

$$2\pi \int_0^{R\phi} (u_s(s))^2 \, ds = 1 \,, \tag{27}$$

and

$$\int_\Sigma u \, d\Sigma = 0 \,. \tag{28}$$

Because of (28), we have

$$\int_\Sigma \|\nabla u\|^2 \, d\Sigma \geqslant \nu_1 \int_\Sigma u^2 \, d\Sigma \,, \tag{29}$$

where ν_1 denotes the first non zero Neumann eigenvalue of the Laplacian on Σ. We claim that there exists a positive constant A such that

$$\int_\Sigma \|\nabla u\|^2 \, d\Sigma \geqslant A(u(R\phi))^2 \tag{30}$$

holds for all $u \in \mathcal{G}$.

For $u \in \mathcal{G}$, we have

$$1 \geqslant 2\pi \int_0^{R\phi} u_s^2(s) \sin(s/R) \, ds = \frac{1}{R} \int_\Sigma \|\nabla u\|^2 \, d\Sigma \,. \tag{31}$$

and

$$\int_\Sigma u^2 \, d\Sigma = 2\pi \int_0^{R\phi} u^2(s) \sin(s/R) \, ds$$

By the Fundamental Theorem of Calculus, we get

$$2\pi u(R\phi) \sin^2(\phi) = 2\pi \int_0^{R\phi} u_s(s) \sin^2(s/R) + \frac{2}{R} u(s) \cos(s/R) \sin(s/R) \, ds \,.$$

From this, we easily obtain

$$\begin{aligned} 2\pi |u(R\phi)| \sin^2(\phi) \;&\leqslant\; 2\pi \Big(\int_0^{R\phi} u_s^2(s) \sin(s/R) \, ds\Big)^{1/2} \Big(\int_0^{R\phi} \sin(s/R) \, ds\Big)^{1/2} \\ &\quad + \frac{4\pi}{R} \Big(\int_0^{R\phi} u^2(s) \sin(s/R) \, ds\Big)^{1/2} \Big(\int_0^{R\phi} \sin(s/R) \, ds\Big)^{1/2} \\ &= R(1-\cos(\phi)\bigg(\Big(\int_\Sigma \|\nabla u\|^2 \, d\Sigma\Big)^{1/2} + \frac{2}{R} \Big(\int_\Sigma u^2 \, d\Sigma\Big)^{1/2} \bigg) \,. \end{aligned}$$

By using (29), the claim easily follows.

The statements (29) and (30) impy that

$$\inf_{u \in \mathcal{G}} \delta_u^2 E > -\infty \,.$$

Let $\{u_n\} \subset \mathcal{G}$ be a minimizing sequence for $\delta_u^2 E$. Using (27), (31), (30) and a standard argument, one sees that the functions in $\mathcal{G}$ are uniformly bounded and equicontinuous. Hence, by the Arzela-Ascoli Theorem, the set of functions $\mathcal{G}$ has compact closure in $C(\Sigma)$. Also, by (29), (31) and the Banach-Alaoglu Theorem, $\mathcal{G}$ has compact closue in $H^1(\Sigma)$ with respect to the weak topology.

We can therefore assume, by passing to a subsequence if necessary, that the sequence $\{u_n\}$ converges weakly in $H^1(\Sigma)$ and strongly in $C(\Sigma)$ to a function $u \in H^1(\Sigma) \cap C(\Sigma)$ which realizes the infimum of $\delta^2 E$ in $\mathcal{G}$. We take the variational derivative of $\delta^2_u E$ with respect to any *compactly supported* $f \in \mathcal{G}$ and get

$$\partial_t(\delta^2_{u+tf}E)_{t=0} = \int_\Sigma \nabla u \cdot \nabla f - \frac{2}{R^2} uf \, d\Sigma = 0$$

This shows that u is a weak solution of $\Delta u + 2R^{-2}u = \text{constant}$. By elliptic regularity, u must be a strong solution. It then follows that $u = c\psi_1$, where ψ_1 is given by (25) and c is a non zero constant. Therefore the minimum of $\delta^2 E$ for functions in $\mathcal{G}$ is zero. From this the result follows in the case that Σ is a spherical cap.

If Σ is the flat horizontal disc of radius R, note that $\cos\alpha = \sqrt{1-R^2}$ and $\sin\alpha = R$, so the second variation formula (11) gives for $\psi = \psi(r)$

$$\delta^2 E = -\int_\Sigma \psi\Delta\psi \, r \, dr d\theta + \oint_{\partial\Sigma} \psi(\psi_r - [\frac{1}{R} + \frac{\beta}{R^4}]\psi) \, R d\theta \, . \tag{32}$$

Analogous to the spherical case , we consider the radial solution of

$$\Delta\psi = \text{constant} \, , \tag{33}$$

with mean value zero. The solution is given, up to a multiplicative constant, by $\psi(r) := r^2 - R^2/2$ which satisfies

$$\frac{\psi_r}{\psi} = \frac{4}{R} \, .$$

Inserting this function in the second integral of (32),we find that Σ is stable only if $\beta \leqslant 3R^3$ holds.

The proof of sufficiency is similar to the spherical case and is left to the reader.
q.e.d

We next consider a numerical example for comparison with the result of the previous theorem.

Let $\Sigma(u, V)$ denote the spherical cap enclosing the volume V and having minimum height u and let $f(u, \omega, \beta, V)$ be its energy as given in (1). Figure 4 shows the graph of the derivative $f_u(u, 0.5, 0.1, 0.44)$. By considering the sign of $f_{uu}(u, 0.5, 0.1, 0.44)$, we can see that for $u_0 \approx 0.066$, $\Sigma(u_0, 0.44)$ is an unstable critical point while for $u = u_1 \approx 0.33$, the cap $\Sigma(u_1, 0.44)$ is stable. In, fact Theorem (3.2) gives

$$\beta_{\max}(\Sigma(u_0, 0.44)) \approx 0.003002 < 0.1 < 1.151287375 \approx \beta_{\max}(\Sigma(u_1, 0.44))$$

which verifies the stability, (resp. instability) of $\Sigma(u_1, 0.44)$, (resp. $\Sigma(u_0, 0.44)$).

The corresponding spherical caps are shown in Figure 5.

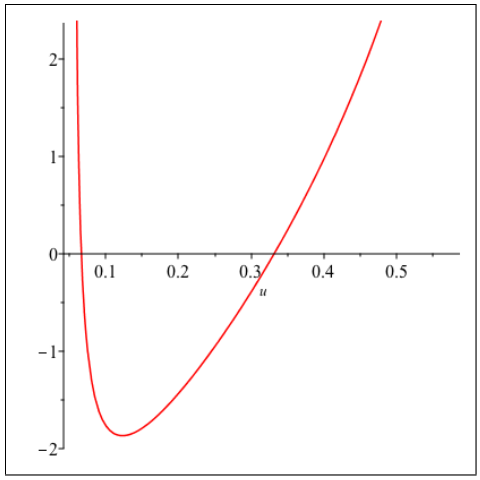
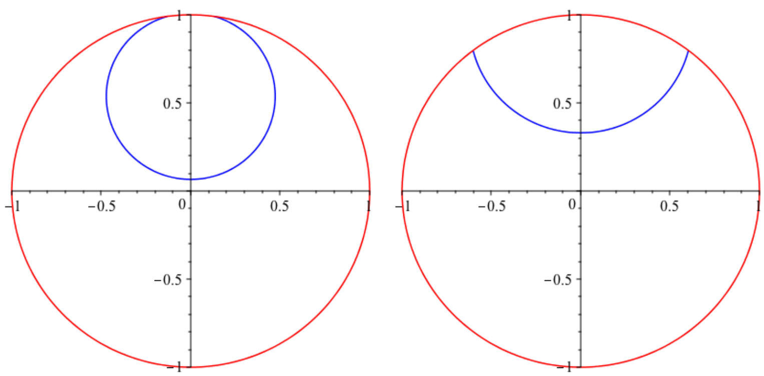


FIGURE 4. Plot of the derivative of the energy $f_u(u, 0.5, 0.1, 0.44)$ as a function of the lowest point u



(A) $\Sigma(u_0, 0.44)$ (B) $\Sigma(u_1, 0.44)$

FIGURE 5

4. APPENDIX: THE SECOND VARIATION FORMULA

We consider a one parameter family of embeddings

$$\begin{array}{rcl} I \times (\Sigma, \partial\Sigma) & \to & (B^3, S^2) \\ (t, p) & \mapsto & \mathbf{X}_t(p)\,, \end{array}$$

where $I = (-\epsilon, \epsilon)$ and $\mathbf{X} \equiv \mathbf{X}_0$. The variation field of order n, $\partial_t^{(n)}(\mathbf{X}_t)_{t=0}$, will be denoted by $\delta^n\mathbf{X}$. As before, we express $\delta\mathbf{X}$ in terms of normal and tangential components as $\psi\mathbf{N} + \mathbf{T}$.

It is assumed that the volume between the surface $\mathbf{X}_t(\Sigma)$ and the 2-sphere is constant. In addition to (3), this implies the second order condition

$$0 = \int_\Sigma \delta^2 \mathbf{X} \cdot \mathbf{N}\, d\Sigma + \int_\Sigma \delta \mathbf{X} \cdot \delta(\mathbf{N} d\Sigma)\,. \tag{34}$$

Using this and the fact that $H \equiv$ constant, we compute the variation of the surface integral in (4).

$$-\delta \int_\Sigma 2H\delta\mathbf{X}\cdot\mathbf{N}\, d\Sigma = -\int_\Sigma 2(\delta H)\delta\mathbf{X}\cdot\mathbf{N}\, d\Sigma = -\int_\Sigma \psi L[\psi]\, d\Sigma\,.$$

We have used here that $\delta H = (1/2)L[\psi] + \nabla H \cdot \mathbf{T} = L[\psi]$ since the mean curvature is constant when $t = 0$. We can then deduce from (9) that, since the term in square brackets vanishes when $t = 0$, the second variation of energy is given by

$$\begin{aligned} \delta^2 E &= -\int_\Sigma \psi L[\psi]\, d\Sigma + \oint_{\partial\Sigma} \delta\big[\mathbf{X}\cdot\mathbf{N} + \beta\bar{k}_g - \omega\big]\mathbf{X}' \times \mathbf{X}\cdot\delta\mathbf{X}\, ds \\ &= -\int_\Sigma \psi L[\psi]\, d\Sigma - \oint_{\partial\Sigma} \delta\big[\mathbf{X}\cdot\mathbf{N} + \beta\bar{k}_g\big]\bar{\mathbf{n}}\cdot\delta\mathbf{X}\, ds \end{aligned} \tag{35}$$

It is clear that the $\mathbf{T}$ does not contribute to the variation of the surface integral (4) and it is also clear from (5) and (6) that the component $\delta\mathbf{X}\cdot\mathbf{X}'$ does not contribute to the variations of the energy E. We can therefore assume that

$$\delta\mathbf{X}\cdot\mathbf{X}' \equiv 0 \tag{36}$$

holds on $\partial\Sigma$ which simplifies the calculations.

Along $\partial\Sigma$ we have two right handed bases $\mathbf{X}, \mathbf{X}', \bar{\mathbf{n}}$ and $\mathbf{n}, \mathbf{X}', \mathbf{N}$. We can therefore write $\mathbf{X} = \cos\alpha\,\mathbf{N} + \sin\alpha\,\mathbf{n}$, $\bar{\mathbf{n}} = \sin\alpha\,\mathbf{N} - \cos\alpha\,\mathbf{n}$. From $\delta\mathbf{X}\cdot\mathbf{X} \equiv 0$, we have $\psi\cos\alpha + \mathbf{T}\cdot\mathbf{n}\sin\alpha \equiv 0$ and we can define

$$\hat{\psi} = \begin{cases} \psi\csc\alpha, & \text{if } \sin\alpha \neq 0, \\ -\mathbf{T}\cdot\mathbf{n} & \text{if } \sin\alpha = 0\,. \end{cases} \tag{37}$$

Using (36), the first order change in the frame $\mathbf{X}, \mathbf{X}', \bar{\mathbf{n}}$ is given by

$$\delta\mathbf{X} = \hat{\psi}\bar{\mathbf{n}},\ \delta\mathbf{X}' = \hat{\psi}'\bar{\mathbf{n}} - \hat{\psi}\bar{k}_g\mathbf{X}',\ \delta\bar{\mathbf{n}} = -\hat{\psi}\mathbf{X} - \hat{\psi}'\mathbf{X}'\,.$$

The first variation of $\bar{k}_g$ is then given by

$$\begin{aligned} \delta\bar{k}_g &= -\delta\Big(\frac{\bar{\mathbf{n}}'\cdot\mathbf{X}'}{\mathbf{X}'\cdot\mathbf{X}'}\Big) \\ &= -(\delta\bar{\mathbf{n}})'\cdot\mathbf{X}' - \bar{\mathbf{n}}'\cdot\ \delta\mathbf{X}' + 2\delta\mathbf{X}'\cdot\mathbf{X}' \\ &= \hat{\psi}'' + \hat{\psi}(1 + \bar{k}_g^2)\,. \end{aligned}$$

Here we have used that when $t = 0$, s is the arc length parameter along $\partial\Sigma$. We also need

$$\begin{aligned} \delta(\mathbf{X}\cdot\mathbf{N}) &= \delta\mathbf{X}\cdot\mathbf{N} + \mathbf{X}\cdot\delta\mathbf{N} \\ &= \psi + \mathbf{X}\cdot(-\nabla\psi + d\mathbf{N}(\mathbf{T})) \\ &= \psi - \psi_n\sin\alpha + (\mathbf{X}\cdot\mathbf{n})((\mathbf{T}\cdot\mathbf{n})d\mathbf{N}(\mathbf{n})\cdot\mathbf{n} \\ &= \psi - \psi_n\sin\alpha - \psi\cos\alpha\, d\mathbf{N}(\mathbf{n})\cdot\mathbf{n}\,. \end{aligned}$$

Combining this with $\delta\mathbf{X}\cdot\bar{\mathbf{n}} = \psi/\sin\alpha$, gives

$$\delta^2 E = -\int_\Sigma \psi L[\psi]\, d\Sigma + \oint_{\partial\Sigma} \psi\big(\psi_n - \psi(\frac{1}{\sin\alpha} - \cot\alpha\, d\mathbf{N}(\mathbf{n})\cdot\mathbf{n})\big)\, ds - \beta\oint_{\partial\Sigma} \hat{\psi}(\hat{\psi}'' + \hat{\psi}(1+\bar{k}_g^2))\, ds\,.$$

Note that at points where $\sin\alpha = 0$, we can make sense of the integrand by replacing $\psi/\sin\alpha$ with $\hat{\psi}$ defined by (37). Also, in the case $\beta = 0$, this formula agrees with that found in [11].

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