Minimal Surfaces with Elastic and Partially Elastic Boundary

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We study equilibrium surfaces for an energy which is a linear combination of the area and a second term which measures the bending and twisting of the boundary. Specifically, the twisting energy is given by the twisting of the Darboux frame.

This energy is a modification of the Euler-Plateau functional considered by Giomi and Mahadevan, [7], and a natural special case of the Kirchhoff-Plateau energy considered by Biria and Fried, [3, 4].

I. INTRODUCTION

Minimal surfaces, which are commonly used to model a thin fluid membrane, are usually considered to have an immovable boundary which is either prescribed (the Plateau problem) or constrained to lie in a fixed supporting surface (the free boundary problem). One exception is the thread problem, where only the length of the boundary is prescribed, [1].

The current paper represents a confluence of ideas, which have been considered previously, concerning minimal surfaces with elastic boundary components. Giomi and Mahadevan, [7] investigate a configuration whose energy consists of the surface tension of a homogeneous membrane coupled to the bending energy of its boundary, the so called Euler-Plateau problem. Shortly afterwards, Biria and Fried, [3, 4] treated the more general case where the boundary curve, regarded as a flexible rod, is allowed to twist, the so called Kirchhoff-Plateau problem. This twisting requires an additional variable in the energy which is a choice of an orthonormal framing of the normal bundle of the curve. Somewhat earlier, [8], Guven et. al. considered a class of elastic energies for curves on surfaces which is more general than that used in [7]. Their formulation does not allow the surface to vary in space, but instead it is considered as an "environment" for the curve.

Here, we will replace the bending energy in the Kirchhoff-Plateau problem by one of the curvature energies discussed in [8]. Alternatively this can be viewed as requiring the framing of the boundary of the minimal surface in the Kirchhoff-Plateau problem to be the Darboux frame consisting of the normal and conormal of the boundary. Equilibrium surfaces for this problem can be interpreted as follows. If we replace the rod in the Kirchhoff- Plateau problem by a twisted ribbon, then an equilibrium surface will meet this ribbon in a right angle, i.e. it will be a solution of the free boundary problem where the supporting surface is the ribbon.

Using an idea due to Nitsche, we show that in the case when the minimal surface is a topological disc, the only solutions are flat planar domains studied in [2]. In the more general situation where the surface may be contained in a solid cylinder $\mathbf{R}^3/\mathbf{Z}$, then, in contrast, the domains in a helicoid bounded by parallel helices provide examples of equilibria with annular topology.

One could just as well consider a minimal surface whose boundary consists of multiple arcs, some of which are fixed and others of which are elastic, which we will refer to as having *partially elastic boundary*. With this in mind, we prove a type of local existence result which produces many examples of minimal surfaces having a boundary arc which satisfies the Euler-Lagrange equations for our variational problem. Modification of this method can also be used to produce local examples for the problems occurring in [7] and [3] discussed above.

II. KIRCHHOFF-PLATEAU VARIATIONAL PROBLEM

Let Σ be a compact, connected surface with boundary and consider the immersion of Σ in the Euclidean 3-space, $\mathbf{R}^3$,

$$X : \Sigma \to \mathbf{R}^3 .$$

Throughout this paper, we assume that $X(\Sigma)$ is an oriented surface of class $\mathscr{C}^2$ immersed in $\mathbf{R}^3$ with piecewise smooth boundary, $\partial\Sigma$, and area

$$\mathscr{A}[X] := \int_\Sigma d\Sigma .$$

Let ν denote a unit normal vector field along Σ.

For a sufficiently smooth curve $C : I \to \mathbf{R}^3$, we denote by $s \in I = [0,L]$ the arc-length parameter of C, where L stands for its length. Then, if $(\,)'$ represents the derivative with respect to the arc-length, the vector field $T(s) := C'(s)$ is the unit tangent to C. Moreover, the (Frenet) curvature of C, κ, is defined by $\kappa(s) = \|T'(s)\|$.

A Kirchhoff elastic rod is a thin elastic rod with circular cross sections and uniform density, naturally straight and prismatic when unstressed and which is being held bent and twisted by external forces and moments acting at its ends alone. Recall that the usual formulation of the variational problem for the Kirchhoff elastic rod is to add to the bending energy of a curve in space, the center line, a second energy

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term which measures the twisting of the rod. Quantitatively, this is measured by taking a framing of the normal bundle of the curve and integrating the square of the norm of its derivative in the normal bundle.

As in [11], we denote the normal frame, referred to as the material frame, by $M := \{M_1, M_2\}$. Therefore, we arrive at the energy of an inextensible Kirchhoff rod:

$$\mathscr{K}\left[(C,M)\right] := \int_C \left(\alpha\kappa^2 + \varpi\|\nabla^\perp M_1\|^2 + \beta\right) ds$$

for suitable real constants α, ϖ and β. The constants α and ϖ are called the *flexural rigidity* and *torsional rigidity* of the rod, respectively ([13]), while the constant β serves as a Lagrange multiplier which fixes the length of the rod. Obviously, the case $\varpi = 0$ represents an non-shearable rod and, hence, it reduces to the classical bending energy of the curve C.

The energy of a homogeneous fluid membrane bounded by an elastic curve is obtained ([7]) by adding a multiple of the surface area to the bending energy of the curve resulting in the *Euler-Plateau energy*. When the twisting energy of the rod is included, the resulting functional is called the *Kirchhoff-Plateau energy* ([3])

$$\mathscr{K}\mathscr{P}[(X,M)] = \sigma\mathscr{A}[X] + \mathscr{K}[(X|_{\partial\Sigma}, M)]\,.$$

In order to reach an equilibrium configuration for this problem, the normal frame can be varied independently of the boundary curve. The Euler-Lagrange equation characterizing a critical frame for a given surface with boundary is the simple equation $\nabla^\perp M_1 \cdot M_2 \equiv$ constant, the constant representing a torsion of the normal frame. Apart from this, there is a necessary condition on the boundary curve which can be obtained by varying the surface. The resulting Euler-Lagrange equation for the boundary curve gives a generalization of the equation of a center line for the Kirchhoff rod. The modification appears as a consequence of the interaction of the surface with the boundary curve in attempting to decrease energy.

In the model that we will study here, we consider the case of the Kirchhoff-Plateau energy when the choice of the normal frame is determined by the surface geometry. Specifically, we consider the case where the normal frame is the normal part of the Darboux frame, consisting of the normal ν to the surface and the conormal $n := T \times \nu$ of the boundary. The Darboux frame of $\partial\Sigma$ is the orthonormal frame $\{n, T, \nu\}$. The derivative of this frame with respect to the arc-length parameter s is given by

$$\begin{pmatrix} n \\ T \\ \nu \end{pmatrix}' = \begin{pmatrix} 0 & -\kappa_g & \tau_g \\ \kappa_g & 0 & \kappa_n \\ -\tau_g & -\kappa_n & 0 \end{pmatrix} \begin{pmatrix} n \\ T \\ \nu \end{pmatrix}, \qquad (1)$$

where the functions involved, κ_g, κ_n and τ_g are, respectively, the geodesic curvature, the normal curvature and the geodesic torsion of the boundary relative to the immersion X. In particular, it is clear from the definition that $\kappa^2(s) = \|T'(s)\|^2 = \kappa_g^2(s) + \kappa_n^2(s)$.

In this setting, our Kirchhoff-Plateau energy functional $\mathscr{K}\mathscr{P}[(X,M)]$ for the immersion $X : \Sigma \to \mathbf{R}^3$ is the *potential energy* ($E \equiv E_{\sigma,\alpha,\varpi,\beta}$)

$$E[X] := \sigma\int_\Sigma d\Sigma + \int_{\partial\Sigma}\left(\alpha\kappa^2 + \varpi\|\nabla^\perp\nu\|^2 + \beta\right) ds$$
$$= \sigma\int_\Sigma d\Sigma + \int_{\partial\Sigma}\left(\alpha\kappa^2 + \varpi\tau_g^2 + \beta\right) ds\,.$$

For convenience, we assume that all connected components of the boundary are made of the same material, so that the flexural and torsional rigidities, α and ϖ, and the Lagrange multiplier restricting the length, β, are the same constants for all boundary components.

One way to visualize the choice of the potential energy is to replace the boundary rod with a boundary ribbon. As the curve, which is the center line of the ribbon, and the surface vary so as to decrease the total energy, the surface is required to meet the ribbon in a right angle. Then, it is clear that the unit normal to the ribbon is parallel to the conormal of the boundary $\partial\Sigma$ and, hence, the energy of the ribbon depends on ν obtaining $E[X]$ for the Kirchhoff-Plateau energy.

We will now compute the first variation of the functional $E[X]$. Since we would like to include the case of partially elastic boundary, we will denote by $C = \cup_i C_i$ the union of non fixed arcs, C_i, of $\partial\Sigma$. Consider arbitrary variations of the immersion $X : \Sigma \to \mathbf{R}^3$, i.e. $X + \varepsilon\delta X + \mathscr{O}(\varepsilon^2)$. By the considerations above, we require that $\delta X \equiv 0$ on $\partial\Sigma \setminus C$. On the other hand, on C, we will denote the restriction of δX by δC. Then, for each term in the energy, we have the following variation formulas:

- **The area functional:**
  The first variation of the area for an arbitrary variation δX is given by

  $$\delta\left(\int_\Sigma d\Sigma\right) = -2\int_\Sigma H\nu\cdot\delta X\, d\Sigma + \int_C n\cdot\delta C\, ds\,,$$

  where H is the mean curvature of Σ.

- **The bending energy:**
  Using the standard formula for the variation of the (Frenet) curvature, κ,

  $$\delta\kappa = \frac{1}{\kappa}(\delta C)''\cdot T' - 2\kappa(\delta C)'\cdot T$$

  and that $\delta(ds) = (\delta C)'\cdot T\, ds$, we get after integrating by parts

  $$\delta\left(\int_C \kappa^2\, ds\right) = \int_C \left(2T'' + 3\kappa^2 T\right)'\cdot\delta C\, ds$$
  $$+ \left(2T'\cdot[\delta C]' - \left[2T'' + 3\kappa^2 T\right]\cdot\delta C\right|_{\partial C}\,,$$

  where T is the unit tangent to the boundary and its derivatives with respect to the arc-length parameter s are given in (1).

- **The twisting energy:**
  As computed in Appendix A, the first variation of the

twisting energy is

$$\delta\left(\int_C \tau_g^2\, ds\right) = \int_C \left(2\tau_g'\left[\kappa_n - 2H\right] n + \tau_g^2 T' + 2\left[\tau_g T \times T'\right]'\right)\cdot \delta C\, ds - 2\int_C \tau_g' \partial_n \left(\nu\cdot \partial C\right) ds + \left(2\tau_g \partial_n\left[\nu\cdot\delta C\right] - \left[2\tau_g\left(\kappa_n - 2H\right) n + 2\tau_g T\times T' - \tau_g^2 T\right]\cdot \delta C\right|_{\partial C},$$

where ∂_n means derivative in the conormal direction.

- **The length functional:**
  Using an argument involving integration by parts, we have that

$$\delta\left(\int_C ds\right) = -\int_C T'\cdot \delta C\, ds + \left(T\cdot \delta C\,\right|_{\partial C},$$

where we have used once again the standard formula $\delta(ds) = (\delta C)'\cdot T ds$.

If we define a vector field along $\partial\Sigma$ as follows,

$$J := 2\alpha T'' + \left(3\alpha\kappa^2 + \varpi\tau_g^2 - \beta\right) T + 2\varpi\tau_g T\times T', \qquad (2)$$

then, combining the information above, we can express the first variation formula for the potential energy $E[X]$, by

$$\delta E[X] = -2\sigma\int_\Sigma H\nu\cdot\delta X\, d\Sigma + \int_C \left(J' + \sigma n + 2\varpi\tau_g' d\nu\left(n\right)\right)\cdot \delta C\, ds - 2\varpi\int_C \tau_g'\partial_n\left[\nu\cdot\delta C\right] ds + \left(2\alpha T'\cdot\left[\delta C\right]' + 2\varpi\tau_g\left[\partial_n\left(\nu\cdot\delta C\right) - d\nu\left(n\right)\cdot\delta C\right] - J\cdot\delta C\,\right|_{\partial C}.$$

For the configuration to be in equilibrium, it is clear by considering compactly supported variations, that $H\equiv 0$ must hold, i.e. the immersion X is minimal. Next, by taking normal variations $\delta X = \psi\nu$ with ψ vanishing near ∂C, we obtain the boundary integral

$$0 = \int_C \left(J'\cdot\psi\nu - 2\varpi\tau_g'\partial_n\psi\right) ds.$$

However, ψ and $\partial_n\psi$ can be chosen to be arbitrary functions on the boundary. For example, the biharmonic Dirichlet problem $\Delta^2\psi = 0$ in Σ, with continuous boundary conditions $\psi = f$, $\partial_n\psi = g$ is well posed. From this we conclude that $\tau_g'\equiv 0$, so τ_g is locally constant on the boundary (it may have different constant values in each connected component of the boundary) and

$$J'\cdot\nu \equiv 0 \qquad (3)$$

on the boundary.

By taking variations that are tangential to the immersion, we also deduce the boundary condition

$$J' + \sigma n = 0, \qquad (4)$$

where J has been defined in (2). Note that $\|J'\|^2 = \sigma^2\in\mathbf{R}^+$ is a consequence of this equation. This condition does not make reference to the surface itself.

Using equations (1), (3) and (4), the Euler Lagrange equations for equlibria of $E[X]$ can be summarized:

$$H\equiv 0, \quad \text{on}\,\Sigma, \qquad (5)$$

while on C, the following hold:

$$\tau_g \equiv \text{constant}, \qquad (6)$$

$$2\alpha\kappa_g'' + \left(\alpha\kappa^2 + \left[3\varpi - 2\alpha\right]\tau_g^2 - \beta\right)\kappa_g - 2\left(2\alpha - \varpi\right)\tau_g\kappa_n' + \sigma = 0, \qquad (7)$$

$$2\alpha\kappa_n'' + \left(\alpha\kappa^2 + \left[3\varpi - 2\alpha\right]\tau_g^2 - \beta\right)\kappa_n + 2\left(2\alpha - \varpi\right)\tau_g\kappa_g' = 0. \qquad (8)$$

These equations are a modified version of a system appearing in [8]. Since κ can be expressed in terms of the geodesic and normal curvatures using $\kappa^2 = \kappa_g^2 + \kappa_n^2$, the Euler-Lagrange equations (7) and (8) have only the unknowns $\kappa_g(s)$ and $\kappa_n(s)$.

When the boundary is regarded as a twisted rod ($\varpi\neq 0$) made of homogeneous isotropic material, the coefficients α and ϖ are related by $\alpha = (1+\varepsilon)\varpi$, where the constant ε denotes the *Poisson's ratio*, [6]. This constant ε, which is a property of the boundary material itself, may vary from -1 to $1/2$. However, the negative values are only obtained for rare materials usually linked with anisotropy, namely, auxetic materials, [5]. With one exception in Chapter 4, we are only going to consider $\varepsilon\in[0,1/2]$, i.e. for any $\varpi > 0$, $\alpha\in[\varpi, 3\varpi/2]$, although most of our results are also true for arbitrary values of α and ϖ. Note that the restrictions on α and ϖ clarify some of the coefficients appearing in (7) and (8).

Equation (7) results from considering only variations which are tangent to the minimal surface. Thus, if C is a closed curve lying on a minimal surface which satisfies this equation, it represents a critical point for the problem of minimizing the elastic energy of the curve with constrained enclosed area. The presence of the constant σ contains the interaction between the surface and curve geometries, [8].

Equation (8) is obtained by taking only normal variations of the minimal surface. Somewhat surprisingly, the condition (6) also arises from considering only normal variations. It is worth noting that we could also obtain the condition (6) by demanding that the normal part of the Darboux frame,

$\{\nu, n\}$, is a critical frame in the Kirchhoff-Plateau problem $\mathscr{K}\mathscr{P}[(X,M)]$.

Note that we have a first integral

$$\left[\left(\kappa_g'\right)^2+\left(\kappa_n'\right)^2\right]+\frac{\alpha}{2}\kappa^4 \\ +\left([3\varpi-2\alpha]\,\tau_g^2-\beta\right)\kappa^2+2\sigma\kappa_g\equiv\text{constant}.$$

The system consisting of (6), (7) and (8) has fixed points located at $(\kappa_g,\kappa_n)=(\bar{\kappa}_g,0)$, where $\bar{\kappa}_g$ is a real solution of the cubic $\alpha x^3+([3\varpi-2\alpha]\,\tau_g^2-\beta)x+\sigma=0$. When $\tau_g=0$, these solutions correspond to circles, which bound flat discs, while for $\tau_g\neq 0$, the solutions give rise to helices, which bound helicoids as will be discussed below. By using the first integral to construct a Lyapunov function, we see that the fixed points are stable (but not asymptotically stable), when $\alpha\bar{\kappa}_g^2+([3\varpi-2\alpha]\,\tau_g^2-\beta)>0$ holds.

If $X:\Sigma\to\mathbf{R}^3$ is a minimal immersion of Σ in $\mathbf{R}^3$, there locally exists a conjugate isometric minimal immersion, Y, defined by the condition that $X+iY$ defines a holomorphic curve in $\mathbf{C}^3$, $\mathbf{C}$ denoting the complex plane. Considering our choice of orientation along the boundary, this amounts to the Cauchy-Riemann equation

$$(\partial_n+i\partial_s)(X+iY)=0\,,$$

along $\partial\Sigma$. In particular, from the real part of this equation, we can assume that $n\equiv\partial_s Y=Y'$ holds on $\partial\Sigma$. Then, using this in the boundary condition (4) we obtain a third order conservation law which holds on $\partial\Sigma$,

$$J+\sigma Y\equiv\Lambda\,, \tag{9}$$

for some constant vector $\Lambda\in\mathbf{R}^3$.

Remark. Observe that if C represents any entire closed connected component of the boundary $\partial\Sigma$, then integrating the boundary condition (4) along C, we get that

$$\oint_C n\,ds=0\,.$$

If C is the entire boundary, this identity can also be derived by taking the first variation of area with respect to constant vector fields on $\mathbf{R}^3$.

The boundary condition gives us a way of computing the area of the minimal surface whose boundary verifies (4). Indeed, it turns out that this area is completely determined by data of the boundary, as the following result shows.

Proposition II.1 *Assume there exists a minimal surface Σ such that the boundary condition* (4) *is satisfied and denote by $C=\cup_i C_i$ the union of the connected components of $\partial\Sigma$. Then the following relation holds:*

$$2\|J'\|\mathscr{A}[\Sigma]+\sum_i\left(\beta-\varpi\tau_g^2\right)\mathscr{L}[C_i]=\alpha\oint_C\kappa^2\,ds$$

where $\mathscr{L}$ is the length functional and J is defined in (2).

Proof. Consider the potential energy $E[X]$, and make a variation by rescalings, i.e. $X\to rX$ for $r>0$. The area rescales quadratically, the length linearly, while the other term rescales like r^{-1}. That is,

$$E[rX]=\sigma r^2\int_\Sigma d\Sigma+\frac{1}{r}\oint_{\partial\Sigma}\left(\alpha\kappa^2+\varpi\tau_g^2\right)ds+\beta r\oint_{\partial\Sigma}ds\,.$$

Thus, differentiating with respect to r, we get at the critical point $r=1$ that the following relation must hold

$$0=2\sigma\mathscr{A}[\Sigma]+\beta\mathscr{L}[\partial\Sigma]-\oint_{\partial\Sigma}\left(\alpha\kappa^2+\varpi\tau_g^2\right)ds\,.$$

Finally, if we apply that τ_g is locally constant along C (recall that it may have different constant values in each connected component C_i) and that σ is, precisely, $\|J'\|$ as mentioned above, we get the statement. **q.e.d.**

III. A LOCAL EXISTENCE THEOREM

Let $C(s)$ be an arc-length parameterized smooth curve in $\mathbf{R}^3$. We denote the usual Frenet frame along C by $\{T,N,B\}$, where N and B are the unit normal and unit binormal to C, respectively. If C is a geodesic, i.e. a straight line, it should be understood that N and B are any unit orthogonal constant vector fields of the normal bundle to C, $\perp C$. Note that for the Kirchhoff-Plateau problem defined in $E[X]$ with no fixed arcs, each connected component of the boundary (represented by C_i) must be closed curves. Therefore, they cannot be geodesics, i.e. their (Frenet) curvature, κ, is non-zero. Consequently, in this case, the Frenet frame is well defined along C_i. Moreover, the closure condition of C_i also implies that both the curvature, $\kappa(s)$, and the (Frenet) torsion, $\tau(s)$, are periodic functions.

In general, the Frenet equations involving the curvature, κ, and torsion, τ, of a curve $C(s)$ are given by

$$\begin{pmatrix}T\\N\\B\end{pmatrix}'=\begin{pmatrix}0&\kappa&0\\-\kappa&0&\tau\\0&-\tau&0\end{pmatrix}\begin{pmatrix}T\\N\\B\end{pmatrix}, \tag{10}$$

where, again, $()'$ means derivative with respect to the arc-length parameter s.

Denote by θ the angle between the normal to the surface Σ, ν, and the normal to the non fixed boundary C, N. This angle θ will be referred to as the *contact angle* between the surface and the boundary. (This angle is actually the contact angle between the surface and the ruled, developable surface given by $(s,t)\mapsto T(s)+tN(s)$.)

Then, using complex coordinates in the normal bundle of C, $\perp C$, we have that

$$\nu+in=e^{i\theta}\,(N+iB)\,. \tag{11}$$

Here, we are assuming that when listing the vectors in any orthonormal frame of $\perp C$, the frame has the same orientation as the Frenet frame $\{N,B\}$. Now, a simple calculation involving differentiation of above equation together with formulae (1) and (10), shows that κ_g, κ_n and τ_g are related with κ, τ and θ by the following equations

$$\kappa_g = \kappa \sin\theta\,, \tag{12}$$

$$\kappa_n = \kappa \cos\theta\,, \tag{13}$$

$$\tau_g = \theta' - \tau\,. \tag{14}$$

Consider now the system of equations for unknowns $\kappa_g(s)$ and $\kappa_n(s)$, given by the boundary conditions (7) and (8), where α, β, ϖ, σ and τ_g are regarded as constants. Short time existence for this system with prescribed first order initial conditions is standard. Note that the right hand side is real analytic in κ_g, κ_g', κ_n, κ_n' and thus all solutions of this system are real analytic [10].

In the following theorem we give a way of constructing a local equilibrium configuration for the potential energy $E[X]$.

Theorem III.1 *Let $\kappa_g(s)$, $\kappa_n(s)$ be a solution of the system* (7)-(8)*, having no common zeros in an interval $I \subset \mathbf{R}$. Then there exists an arc-length parameterized curve $C(s)$, $s \in I$, and a minimal surface with boundary $X : \Sigma \to \mathbf{R}^3$ such that the image of C is contained in the boundary of Σ and the geodesic curvature, normal curvature and geodesic torsion of C relative to the surface are $\kappa_g(s)$, $\kappa_n(s)$ and τ_g, respectively.*

Proof. Assume that $\kappa_g(s)$ and $\kappa_n(s)$ are solutions of (7)-(8) in the conditions of the statement. On the interval I, we define

$$\kappa(s) := \sqrt{\kappa_g^2(s) + \kappa_n^2(s)}$$

and an angle $\theta(s)$ by the equations (12) and (13). Now, we use equation (14) to define a torsion, $\tau(s)$. Then, both $\kappa(s)$ and $\tau(s)$ are real analytic functions of s. By the Fundamental Theorem of Curves, there exists a curve $C(s)$ defined on I. Since C is found by integrating the Frenet equations, which have real analytic coefficients, the curve $C(s)$ is also real analytic on I. This follows by again using results of [10].

We next define a unit vector field orthogonal to $C'(s)$ and making an angle $\theta(s)$ with the Frenet normal $N(s)$, $\nu(s)$. By analyticity, the curve C and the vector field ν, defined along it, have holomorphic extensions $C(z)$ and $\nu(z)$, for a complex variable $z = s + it$, to a simply connected domain U in the complex plane $\mathbf{C}$, with $I \subset U$. Thus, for fixed $s_o \in I$, we use the following Bjorling's Formula

$$X(z) := \Re\left(C(z) + i\int_{s_0}^{z} C'(\omega)\times\nu(\omega)\,d\omega\right)$$

to construct a minimal surface, containing the curve C, whose unit normal along C is, precisely, $\nu(s)$. We chose the conormal to $C(s)$, relative to the surface, to be $n(s) := C'(s)\times\nu(s)$, then equation (11) holds along I. Differentiating this formula and using equations (1) and (10), we obtain

$$-\left(\widetilde{\kappa}_n + i\widetilde{\kappa}_g\right)C' + i\widetilde{\tau}_g\left(\nu + in\right) = -\kappa e^{i\theta}C' + i\left(\theta' - \tau\right)e^{i\theta}\left(N + iB\right),$$

where $\widetilde{\kappa}_g$, $\widetilde{\kappa}_n$ and $\widetilde{\tau}_g$ are, respectively, the geodesic curvature, normal curvature and geodesic torsion of C relative to the minimal surface $X(z)$. We see from the last equations that

$$\kappa e^{i\theta} = \widetilde{\kappa}_n + i\widetilde{\kappa}_g, \qquad \widetilde{\tau}_g = \theta' - \tau,$$

holds and so it follows from (12)-(14), that $\widetilde{\kappa}_n \equiv \kappa_n$, $\widetilde{\kappa}_g \equiv \kappa_g$ and $\widetilde{\tau}_g \equiv \tau_g$ hold.

If we take Σ to be the part of the minimal surface $X(z)$ constructed above which lies on one side of the curve C, we obtain the result. **q.e.d.**

The proof of the previous theorem shows the difficulty with finding global examples of critical points for the functional $E[X]$, i.e. minimal surfaces bounded by a closed curve, or system of closed curves, satisfying (6)-(8), on the boundary. Suppose that a closed curve C satisfying the boundary conditions was found. Then for a global solution, we would need to solve the Plateau problem of finding a minimal surface Σ with boundary C. In addition to this, the angle θ between the surface normal and curve normal would have to satisfy the condition (14). This shows that the problem of producing examples is quite overdetermined.

The construction above can be used to produce examples of minimal surfaces having *partially elastic boundary*. If we take a local solution, i.e. a minimal surface Σ having a non closed boundary arc C satisfying (6)-(8), we can connect the endpoints of C by an arc C^* contained in Σ. We then just regard C^* as the fixed boundary component.

Moreover, the previous theorem can also be used to prove the local existence of critical points for the original Euler-Plateau problem introduced by Giomi and Mahadevan, [7]. Firstly, the boundary condition (4) implies that the potential energy $E[X]$ can be written as a combination of the area functional with a boundary energy defining center lines of Kirchhoff elastic rods, that is,

$$E[X] = \sigma\int_\Sigma d\Sigma + \int_C \left(\alpha\kappa^2 + \mu\tau + \lambda\right)ds\,, \tag{15}$$

where μ and λ are two constants related with the parameters ϖ and β and with the constant geodesic torsion, τ_g, as follows

$$\mu = -2\varpi\tau_g\,, \qquad \lambda = \beta - \varpi\tau_g^2\,.$$

If we rewrite the boundary condition (4) in terms of the Frenet frame $\{T, N, B\}$ along the boundary, we obtain $J' + \sigma n = 0$ where the vector field J is now given by,

$$J = \left(\alpha\kappa^2 + \varpi\tau_g^2 - \beta\right)T + 2\alpha\kappa' N + 2\left(\alpha\tau + \varpi\tau_g\right)\kappa B\,. \tag{16}$$

It turns out that $J' + \sigma n = 0$ is, precisely, the boundary condition of the energy (15). In particular, this new expression of the Euler-Lagrange equations (7)-(8) allows us to rewrite them in terms of the curvature, κ, and torsion, τ, of the boundary, obtaining the following system

$$2\alpha\kappa'' + \left(\alpha\kappa^2 + \varpi\tau_g^2 - \beta\right)\kappa - 2\kappa\tau\left(\alpha\tau + \varpi\tau_g\right) + \sigma\sin\theta = 0\,, \tag{17}$$

$$\left(\kappa^2\left[2\alpha\tau + \varpi\tau_g\right]\right)' + \sigma\kappa\cos\theta = 0\,, \tag{18}$$

where θ is the contact angle.

A first consequence is that if in the second equation in the system above we use equation (13) and integrate over any closed component of the boundary, C, we get that

$$\oint_C \kappa_n\,ds = 0\,. \tag{19}$$

Let $\bar{C}(s)$ be a new curve in $\mathbf{R}^3$ uniquely defined, up to rigid motions, by the following curvature, $\bar{\kappa}$, and torsion, $\bar{\tau}$,

$$\bar{\kappa} = \kappa\,, \qquad \bar{\tau} = \tau + \frac{\varpi}{2\alpha}\tau_g\,.$$

This transformation is a particular case of one for center lines of Kirchhoff rods described in [11]. Later, in [3], it was also used to transform solutions of the general Kirchhoff-Plateau problem to the Euler-Plateau one.

Then, using it, we have the following local existence result for the Euler-Plateau problem of [7].

Corollary III.1 *Let $C(s)$ be the curve of Theorem III.1 with curvature $\kappa(s)$ and torsion $\tau(s)$. Then, locally there exists a curve $\bar{C}(s)$ (defined as above) and a minimal surface with boundary $\bar{X}:\bar{\Sigma}\to\mathbf{R}^3$ such that $\bar{X}$ is a critical point of the energy*

$$\bar{E}[\bar{X}] := \sigma\int_{\bar{\Sigma}} d\bar{\Sigma} + \int_{\bar{C}}\left(\alpha\bar{\kappa}^2+\bar{\beta}\right)ds\,,$$

where $\bar{\beta} = \beta - \varpi\tau_g^2\left(1+\frac{\varpi}{2\alpha}\right)$ and $\bar{\theta}\equiv\theta$.

Proof. By the argument above, it is clear that the new curve $\bar{C}$ is locally a boundary curve satisfying the Euler-Lagrange equations (17) and (18) for the functional $\bar{E}[\bar{X}]$ with respect to the parameters of the statement.

Now, considering once more the Bjorling's Formula for the new data, i.e. the associated data to $\bar{C}(s)$, we get a minimal surface where $\bar{C}$ is its boundary satisfying the Euler-Lagrange equations of $\bar{E}[\bar{X}]$. **q.e.d.**

Note that previous corollary provides local existence of critical immersions for the Euler-Plateau problem $\bar{E}[\bar{X}]$, where the geodesic torsion of the boundary, $\bar{\tau}_g$, is constant. Indeed, from formula (14) for $\bar{X}$, we have, using the definition of $\bar{C}$,

$$\begin{aligned}\bar{\tau}_g &= \bar{\theta}' - \bar{\tau} = \theta' - \left(\tau + \frac{\varpi}{2\alpha}\tau_g\right) = \theta' - \tau - \frac{\varpi}{2\alpha}\tau_g \\ &= \tau_g\left(1-\frac{\varpi}{2\alpha}\right)\end{aligned}$$

which is clearly constant.

IV. GLOBAL RESULTS OF EQUILIBRIUM CONFIGURATIONS

For an equilibrium immersion $X:\Sigma\to\mathbf{R}^3$ of the potential energy $E[X]$, equations (5) and (6) implies that Σ is minimal and that τ_g is constant along the non fixed boundary, respectively. From these properties, we can prove a first result concerning surfaces which are topologically discs, $\Sigma\cong D$. Indeed, adapting an argument due to Nitsche, we have

Theorem IV.1 *Let $X:\Sigma\to\mathbf{R}^3$ be an immersion of a minimal surface of disc type having constant geodesic torsion along the boundary. Then, the surface is a planar domain. In particular, there are no non-planar critical surfaces for the functional $E[X]$ having the topology of the disc.*

Proof. We may assume that the surface is given by a conformal immersion of the unit disc $X:\mathscr{D}\to\mathbf{R}^3$. Let z be the usual complex coordinate in the disc and let $\omega := \log z$. Although ω is not well defined, $d\omega = dz/z$ is well defined in $\mathscr{D}\setminus\{0\}$. In a neighborhood of $\partial\mathscr{D}$, we express the fundamental forms of the immersion as

$$\begin{aligned} ds_X^2 &= e^{\mu}|d\omega|^2\,,\\ \Re\left(\widetilde{\Phi}\,d\omega^2\right) &= \Re\left(-\left[L_{22}+iL_{12}\right]d\omega^2\right)\,,\end{aligned}$$

where L_{ij}, $i,j=1,2$ are the coefficients of the second fundamental form. Here we have used that the surface is minimal, so $L_{11}=-L_{22}$. These functions are known, [9], to satisfy the Codazzi and Gauss equations, which in the case of a minimal surface, state

$$\widetilde{\Phi}_{\bar{\omega}} = 0\,, \qquad |\widetilde{\Phi}|^2e^{-2\mu} = -K\,.$$

The first of these means that $\widetilde{\Phi}$ defines a holomorphic function.

Note that on $\partial\mathscr{D}$, we have

$$\widetilde{\Phi} = -e^{\mu}\left(k_n + i\tau_g\right)\,.$$

We first assume that $\tau_g\equiv 0$ holds on $\partial\mathscr{D}$. We use the transformation law for quadratic differentials, to obtain the following relation between the Hopf differential in the ω and z coordinates,

$$\widetilde{\Phi}\,d\omega^2 = \widetilde{\Phi}\omega_z^2\,dz^2 = \widetilde{\Phi}\left(\frac{1}{z^2}\right)dz^2 =: \Phi dz^2\,.$$

In contrast to $\widetilde{\Phi}$, Φ is globally defined and holomorphic on $\mathscr{D}$, as is $z^2\Phi$. The calculation above shows that $\widetilde{\Phi} = z^2\Phi$ on $\partial\mathscr{D}$, so we conclude that $\Im\left(z^2\Phi\right)\equiv 0$ holds on $\partial\mathscr{D}$. It follows that on $\mathscr{D}$, $\Phi = c/z^2$ holds for a real constant c, which is impossible unless c, and hence Φ vanishes identically. (Nitsche used this argument to prove that the only disc type minimal surfaces in a ball with free boundary on the sphere are flat discs, [15].)

Next, we consider the case where τ_g is a constant different from zero. Consider the image of $\partial\mathscr{D}$ under the map $-\widetilde{\Phi}$. Since the imaginary part of $-\widetilde{\Phi}$ never vanishes, this image is contained in a half plane, so it is clear that the total variation of $\arg\widetilde{\Phi}$ over $\partial\mathscr{D}$ vanishes. We write this as

$$\mathrm{Var}\arg\widetilde{\Phi}\,|_{\partial\mathscr{D}} = 0\,.$$

However, since $\widetilde{\Phi} = z^2\Phi$ on $\partial\mathscr{D}$, we get

$$\begin{aligned} 0 &= \mathrm{Var}\arg\widetilde{\Phi}\,|_{\partial\mathscr{D}} = \mathrm{Var}\arg z^2\Phi\,|_{\partial\mathscr{D}} \\ &= \mathrm{Var}\arg z^2\,|_{\partial\mathscr{D}} + \mathrm{Var}\arg\Phi\,|_{\partial\mathscr{D}} = 4\pi + \mathrm{Var}\arg\Phi\,|_{\partial\mathscr{D}}\,.\end{aligned}$$

Unless $\Phi\equiv 0$ holds, this gives a contradiction, since, by the Argument Principle, the last term is equal to the total number of zeros of Φ (which are the umbilics) in $\mathscr{D}$, counting multiplicities. So, in particular, the last term is non negative.

Finally, if $\Phi\equiv 0$ holds in $\mathscr{D}$, then every point is planar and, hence, the surface is also planar. **q.e.d.**

For constants $a \neq 0$, $b \in \mathbf{R}$ the immersion

$$X : \mathbf{R}^2 \longrightarrow \mathbf{R}^3, \quad (r, \vartheta) \mapsto (r\cos\vartheta, r\sin\vartheta, a\vartheta + b)$$

defines a minimal helicoid. The curves for constant r are helices, for which

$$\kappa_n = 0\,, \quad \tau_g = \frac{-a}{a^2 + r^2}\,, \quad \kappa_g = \frac{\pm r}{a^2 + r^2}$$

hold. Since these quantities are all constant, it is easily checked that for suitable constants α, β, ϖ and σ, the equations (5)-(7) hold. The domain Δ_{01} in the helicoid defined for any constants $r_0 < r_1$ by $r_0 \leq r \leq r_1$ and $0 \leq \theta \leq 2\pi$ will then correspond to a minimal annulus in a quotient $\mathbf{R}^3/\mathbf{Z}$ which is a critical surface of the functional $E[X]$. Alternatively, the domains Δ_{01} in the helicoid are critical for $E[X]$ having partially elastic boundary where the line segments $\vartheta = \vartheta_i$, $i = 0, 1$ are regarded as the fixed boundary components. The left image of Figure 1 shows a critical domain in the helicoid having partially elastic boundary where the fixed boundary is given by the black line segments.

The helicoid, together with a curvature driven boundary energy, has been used to model a particular structure in biological cells called a stacked endoplasmic reticulum, [14, 16]. In this application, the multivalence of the immersion, which results in the stacking of layers, is an essential property.

The helicoid shows the existence of a critical surface for the functional $E[X]$, such that a quotient $\Sigma/\mathbf{Z}$ is an annulus A. In addition, all of the geometry of Σ, i.e. the first two fundamental forms, descend to A. This type of situation is quite common for minimal surfaces since the integrals used to construct a minimal surface may have periods on a non simply connected domain. In fact, the most common way to parameterize a minimal surface is using the Weierstrass representation. Starting with a trio of holomorphic differentials $(h_1, h_2, h_3)dz$ having no common zeros on a planar domain U and satisfying $h_1^2 + h_2^2 + h_3^2 = 0$,

$$X := \Re\left(\int_{z_0} (h_1, h_2, h_3)\,dz\right)$$

defines a minimal immersion into $\mathbf{R}^3$ which is, in general, unless U is simply connected, multivalued. When $U = A$ is an annulus, then the topology of the surface is completely determined by the period

$$p := \Re\left(\int_\gamma (h_1, h_2, h_3)\,dz\right),$$

where γ is any curve representing the generator of $\pi_1(A)$.

For these kind of multivalued immersions we have the following result.

Theorem IV.2 *Let $X : A \to \mathbf{R}^3$ be a multivalued minimal immersion such that the image surface is a critical point of the functional $E[X]$ in $\mathbf{R}^3$ or $\mathbf{R}^3/\mathbf{Z}$. If $\tau_g = 0$ holds on at least one component of ∂A, then the image surface is planar.*

Proof. We first show that if $\tau_g = 0$ holds on one component of ∂A, then it also holds in both boundary components.

Every annulus A is conformally a domain $1 \leq |z| \leq \rho$. We may assume that $\tau_g \equiv 0$ on $|z| = 1$, otherwise, we can replace X with $X \circ f$ where $f(z) := \rho/z$. Write the Hopf differential as $\Re\left(\Phi\,dz^2\right)$ and define $-z^2\Phi := U + iV$. On $|z| = \rho$, we have

$$\text{constant} = \tau_g = -\Im\left[\left(\frac{z}{|z|}\right)^2 \Phi\right] e^{-\mu} = Ve^{-\mu}\,, \qquad (20)$$

while on $|z| = 1$, we have $V \equiv 0$.

The harmonic measure of $|z| = \rho$ is the harmonic function $\omega(r) := \ln(r)/\ln(\rho)$. On $|z| = \rho$, we have $\partial_n\omega = e^{-\mu/2}\partial_r\omega = e^{-\mu/2}(\rho\ln\rho)^{-1} > 0$. We then obtain

$$\begin{aligned}
\int_{|z|=\rho} V\partial_n\omega\,ds &= \int_{\partial A} V\partial_n\omega\,ds \\
&= \int_{\partial A} \omega\partial_n V\,ds + \int_A (V\Delta\omega - \omega\Delta V)\,dA \\
&= \int_{\partial A} \omega\partial_n V\,ds + 0 = \int_{|z|=\rho} \partial_n V\,ds \\
&= \int_{|z|=\rho} -\partial_s U\,ds = 0\,.
\end{aligned}$$

In the last line, we have used the Cauchy-Riemann equations: $(\partial_n + i\partial_s)(U + iV) = 0$. On $|z| = \rho$, V is either non negative or non positive, by (20), while $\partial_n\omega > 0$ holds, so we can conclude that $V \equiv 0$ holds on $|z| = \rho$ also.

Now we have $V \equiv 0$ on ∂A, hence $V \equiv 0$ and so $z^2\Phi \equiv c$ for a real constant c. Note that on either circle C,

$$-\kappa_n = \Re\left[\left(\frac{z}{|z|}\right)^2 \Phi e^{-\mu}\right] = c|z|^{-2}e^{-\mu}\,.$$

However, integrating above equation along C and using (19) we get that $c = 0$ and hence $\Phi \equiv 0$ holds, so the surface is planar. **q.e.d.**

Recall that any multivalued minimal immersion X can be embedded in a one parameter family of multivalued minimal immersions

$$X_\vartheta := \Re\left(e^{i\vartheta}\int_{z_0} X_z\,dz\right).$$

The case $\vartheta = \pi/2$ just gives the conjugate minimal surface Y appearing in (9). Because of (9) and (16) the immersion $Y = X_{\pi/2}$ is always single valued on A since each boundary curve represents the generator of $\pi_1(A)$. If we denote by p_ϑ the period for the immersion X_ϑ, we have

$$\begin{aligned}
p_\vartheta &= \int_{|z|=\rho} \Re\left(e^{i\vartheta}X_z\,dz\right) = \frac{1}{2}\int_{|z|=\rho} e^{i\vartheta}X_z\,dz + e^{-i\vartheta}X_{\bar{z}}\,d\bar{z} \\
&= \frac{1}{2}\int_{|z|=\rho} \cos\vartheta\,(X_z\,dz + X_{\bar{z}}\,d\bar{z}) + i\sin\vartheta\,(X_z\,dz - X_{\bar{z}}\,d\bar{z}) \\
&= \cos\vartheta\int_{|z|=\rho} dX + \sin\vartheta\int_{|z|=\rho} dY = p_0\cos\vartheta + 0\,,
\end{aligned}$$

i.e. $p_\vartheta = p_0\cos\vartheta$. This means that if $p_0 = 0$ holds, then the image surface is always a topological annulus, while if

$p_0 \neq 0$ then as the surface deforms isometrically, the translational period decreases in magnitude to zero, as ϑ varies in $[0, \pi/2)$ but maintains its direction. In any case, we can imagine the surfaces as being a family of annuli in solid cylinders $\mathbf{R}^3/(\mathbf{Z}(cos\vartheta)p_0)$. Roughly, the surface deforms like a helix made of ribbon compressing to a thin cylinder (see Figure 1).

Since our deformation is isometric, the areas of this family of annuli are unchanging, as are the arc lengths of their boundary curves and the geodesic curvatures of the boundary curves. Examining the boundary terms of the values $E[X_\vartheta]$, we see that the only terms that change with ϑ is

$$f(\vartheta) := \int_{\partial A} \left(\alpha \left[\kappa_n^\vartheta\right]^2 + \varpi \left[\tau_g^\vartheta\right]^2 \right) ds\,,$$

where $\kappa_n^\vartheta + i\tau_g^\vartheta := e^{i\vartheta}\left(\kappa_n + i\tau_g\right)$. Using this, we easily compute

$$\frac{d^2 f}{d\vartheta^2}\Big|_{\vartheta=0} = 2\left(\alpha - \varpi\right) \int_{\partial A} \left(\tau_g^2 - \kappa_n^2\right) ds\,. \qquad (21)$$

Thus, we end up with the following necessary condition for the stability.

Proposition IV.1 *A necessary condition for the stability of a critical annulus as described above, is the non negativity of the integral* (21).

By applying Proposition IV.1, it is clear that the helicoidal domains Δ_{01} are thus unstable when $\alpha < \varpi$ holds, i.e. when the Poisson's ratio is negative, [5].

Finally, as a consequence of Theorem IV.1 and Theorem IV.2, if the surface is either topologically a disc or an annulus where the geodesic torsion vanishes on, at least, one of the boundary components, then it must be planar. Hence, using the condition that $K \equiv 0$ along the boundary, we conclude that $\tau_g = \kappa_n = 0$. Moreover, since the boundary cannot be a straight line, from equation (13) we obtain that the contact angle verifies $\theta = \pi/2$.

V. CONSTANT CONTACT ANGLE

Assume that the contact angle θ is constant along any closed boundary component C. Thus, using (13) and (19) we have that

$$0 = \oint_C \kappa_n\, ds = \oint_C \kappa \cos\theta\, ds = \cos\theta \oint_C \kappa\, ds\,.$$

Since $\kappa > 0$ along C, this is only possible if $\theta \equiv \pi/2$.

Using (12)-(14), we obtain along C, $\kappa_g = \kappa$, $\kappa_n = 0$ and $\tau_g = -\tau \in \mathbf{R}$. We use this data in the boundary condition (see equations (7) and (8)) to obtain

$$2\alpha\kappa'' + \left(\alpha\kappa^2 + [3\varpi - 2\alpha]\,\tau^2 - \beta\right)\kappa + \sigma = 0\,, \qquad (22)$$

$$(2\alpha - \varpi)\,\tau\kappa' = 0\,, \qquad (23)$$

along the closed component of the boundary, C.

Due to the restriction of Poisson's ratio, $\alpha \in [\varpi, 3\varpi/2]$ holds. So $\varpi \neq 2\alpha$, and critical points of $E[X]$ with constant contact angle are characterized as follows.

Theorem V.1 *Let $X : \Sigma \to \mathbf{R}^3$ be a critical immersion for the potential energy $E[X]$ and assume that the contact angle, θ, is constant along at least one closed boundary component C. Then, $\theta = \pi/2$ along the entire boundary $\partial\Sigma$ and $X(\Sigma)$ is a planar domain bounded by area-constrained elasticae.*

Proof. As mentioned above if θ is constant along any closed boundary component, C, then necessarily $\theta = \pi/2$. Hence, consider that $\theta \equiv \pi/2$ along C. By the remark above, $\varpi \neq 2\alpha$, so equation (23) simplifies to $\tau\kappa' = 0$.

If the curvature κ is constant, then C is a Frenet helix (κ and τ constant). Moreover, in order to close up, the torsion must vanish and, as a consequence, C is a circle.

On the other hand, if $\tau = 0$, then equation (22) becomes

$$2\alpha\kappa'' + \left(\alpha\kappa^2 - \beta\right)\kappa + \sigma = 0$$

which implies that C is an area-constrained elastica (see [2]). The case with constant curvature can also be included here in an obvious manner.

Finally, since the surface is minimal, we have

$$K = -\kappa_n^2 - \tau_g^2 = 0$$

along the boundary component C, which, together with the minimality condition, implies that C is a curve of umbilic points. Therefore, $X(\Sigma)$ must be a planar domain bounded by $\partial\Sigma$. Clearly, in the component C the statement holds. For the other boundary components, we have from the flat condition that $\kappa_n = \tau_g = 0$ and, hence, equation (22) becomes the Euler-Lagrange equation of an area-constrained elastica, as before, proving the statement. **q.e.d.**

In particular, if the surface Σ happens to be a topological disc, we get the following result.

Corollary V.1 *Let $X : \Sigma \to \mathbf{R}^3$ be an immersion of a disc type surface critical for $E[X]$, then $X(\Sigma)$ is a compact domain in the plane bounded by an area-constrained elastica. (These domains were studied in detail in [2].)*

Proof. The result follows directly from Theorem IV.1 and Theorem V.1. **q.e.d.**

We point out that in Theorem V.1 we have only used the periodicity of the boundary component C, therefore a similar argument may be used to prove the following result.

Proposition V.1 *Let $X : A \to \mathbf{R}^3$ be a multivalued minimal immersion such that the image surface is a critical point of the functional $E[X]$ in $\mathbf{R}^3/\mathbf{Z}$ and assume that the contact angle, θ, is constant along ∂A. Then, $\theta = \pi/2$ and, if $X(A)$ is not planar, it is a domain bounded by Frenet helices.*

Proof. Let C denote any connected component of the boundary ∂A. Since C is periodic and θ is constant along it, we have the following relation

$$0 = \int_C \kappa_n\, ds = \cos\theta \int_C \kappa\, ds\,.$$

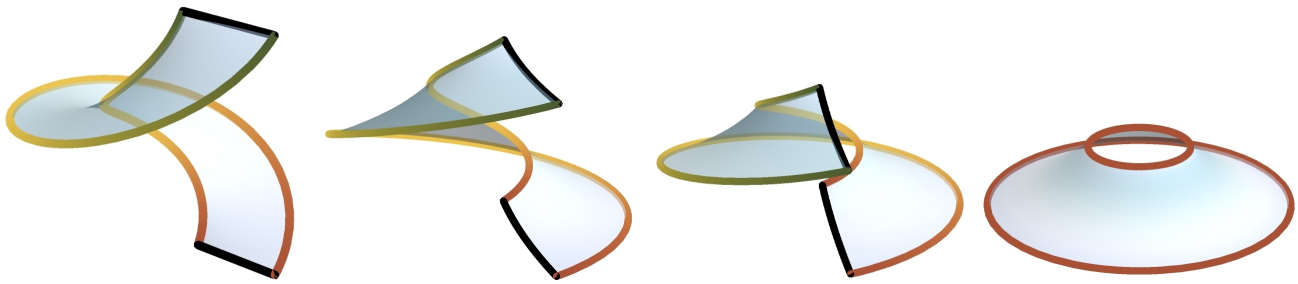

FIG. 1. Four steps of the deformation of a domain of surfaces of the one parameter family of multivalued minimal immersions associated to the helicoid. From left to right: a part of the helicoid ($\vartheta = 0$), $\vartheta = \pi/3$, $\vartheta = 2\pi/5$ and a part of the conjugate surface, i.e. the catenoid ($\vartheta = \pi/2$).

By the same argument as before, we have that this is only possible when $\theta \equiv \pi/2$. In this case, along C, $\kappa_g = \kappa$, $\kappa_n = 0$ and $\tau_g = -\tau \in \mathbf{R}$. In particular, equations (22) and (23) also hold.

If $\tau_g = 0$, we can use Proposition IV.2 to conclude that the domain $X(A)$ is planar. As a consequence, $\theta = \pi/2$ along ∂A.

On the other hand, if $\tau_g = -\tau \neq 0$, we conclude from (23) that κ is constant, since $\varpi \neq 2\alpha$. Then, it is clear that C is a Frenet helix. We then use a similar argument on the other boundary component, proving the result. **q.e.d.**

VI. THE INTRINSIC VARIATIONAL PROBLEM

In this section we fix a minimal surface $X : \Sigma \to \mathbf{R}^3$ and seek a critical curve C in the surface for an extrinsic elastic energy which encloses a relatively compact domain Δ of the surface of prescribed surface area. Specifically, we consider the energy

$$F[C] := \sigma \mathscr{A}[\Delta] + \int_C \left(\alpha\kappa^2 + \varpi\tau_g^2 + \beta\right) ds\,.$$

Here, the curve C may be the entire boundary of the domain Δ or it may only be a union of open arcs of the boundary. As before, in the latter case, we will assume that $\partial\Delta \setminus C$ is kept fixed under deformations.

Since the minimal surface Σ is fixed from the beginning, this problem is equivalent to the problem of finding critica of $E[X]$, when only variations δX tangential to the immersion X are considered. On the non fixed part of the boundary, C, we can write $\delta C = \varphi n + \phi T$ for some compactly supported functions. For these type of variations, we obtain from the first variation formula (Section II)

$$\delta E[X] = \delta F[C] = \int_C \left(J' \cdot n + 2\varpi\tau_g'\kappa_n + \sigma\right) \varphi\, ds = 0\,.$$

Thus, it is clear that $J' \cdot n + 2\varpi\tau_g'\kappa_n + \sigma = 0$ must hold on C. This equation can be rewritten as

$$\begin{aligned} &2\alpha\kappa_g'' + \left(\alpha\kappa^2 + [3\varpi - 2\alpha]\,\tau_g^2 - \beta\right)\kappa_g \\ &-2\left([2\alpha - \varpi]\,\tau_g\kappa_n' - [2\varpi - \alpha]\,\tau_g'\kappa_n\right) + \sigma = 0\,. \end{aligned} \qquad (24)$$

Using the equation above instead of (7), a similar argument to that of Corollary III.1 can also be used to obtain solutions of the original Euler-Plateau problem with non-constant $\bar{\tau}_g$.

Corollary VI.1 *Let $\kappa_g(s)$, $\kappa_n(s)$ be a solution of the system* (8)-(24)*, where τ_g is any analytic function. Then, locally there exists a curve $\bar{C}(s)$ and a minimal surface with boundary $\bar{X} : \bar{\Sigma} \to \mathbf{R}^3$ such that $\bar{X}$ is a critical point of*

$$\bar{E}[\bar{X}] := \sigma \int_{\bar{\Sigma}} d\bar{\Sigma} + \int_{\bar{C}} \left(\alpha\bar{\kappa}^2 + \bar{\beta}\right) ds\,,$$

where $\bar{\beta} = \beta - \omega\tau_g^2\left(1 + \frac{\omega}{2\alpha}\right)$.

Proof. As in the proof of Theorem III.1, from κ_g and κ_n define analytic functions $\kappa(s)$ and $\theta(s)$. Use (14) to define $\tau(s)$ for the arbitrary analytic function $\tau_g(s)$. Now, using the transformations above to get $\bar{\kappa}(s)$ and $\bar{\tau}(s)$, we obtain an analytic curve $\bar{C}(s)$, which verifies the boundary conditions of $\bar{E}[\bar{X}]$.

We finally use the Bjorling's Formula as before to conclude the proof. **q.e.d.**

Note that it may also be used to prove the local existence of critical points of the general Kirchhoff-Plateau problem, using the transformation of the curvature and torsion of [11] and [3].

In particular, if $\varpi = \alpha$, the energy $F[C]$ can be rewritten in an intrinsic form. Physically, this relation between the coefficients of flexural and torsional rigidities means that the material of the boundary rod has zero Poisson's ratio, i.e. it does not overcome lateral expansion when compressed. An example of this kind of materials is cork. Using that for a minimal surface, the Gaussian curvature, K, on the boundary is given by $K = -\kappa_n^2 - \tau_g^2$, and recalling that $\kappa^2 = \kappa_g^2 + \kappa_n^2$, we obtain.

$$\widetilde{F}[C] := \sigma \mathscr{A}[\Delta] + \int_C \left(\alpha\left[\kappa_g^2 - K\right] - \beta\right) ds\,.$$

This variational problem will be referred as the *intrinsic Kirchhoff-Plateau problem*. Now, all quantities appearing in the energy are intrinsically determined on the surface. We recall the the metrics appearing on minimal surfaces have the intrinsic characterization known as the Ricci Condition. This states that away from their flat points, the metric $\sqrt{-K}ds_X^2$ is flat. At the flat points, which are isolated, an additional condition must be imposed [12]. Thus, when $\alpha = \varpi$ holds, the problem can be formulated in an abstract way.

In this case, the Euler-Lagrange equation, (24), reduces to

$$\begin{aligned} 2\alpha\kappa_g'' &+ \left(\alpha\left[\kappa_g^2 - K\right] - \beta\right)\kappa_g \\ &-2\alpha\left(\tau_g\kappa_n' - \tau_g'\kappa_n\right) + \sigma = 0 \,. \end{aligned} \quad (25)$$

Moreover, since our surface is minimal, as computed in [8], we have that along the boundary

$$2\alpha\left(\kappa_n\tau_g' - \kappa_n'\tau_g\right) = 4\alpha\kappa_g K - 2\alpha\nabla K\cdot n$$

holds. Hence, the Euler-Lagrange equation (25) can be rewritten intrinsically as

$$2\alpha\kappa_g'' + \left(\alpha\left[\kappa_g^2 - K\right] - \beta\right)\kappa_g + 4\alpha\kappa_g K = 2\alpha\nabla K\cdot n \,.$$

Finally, as an illustration of the intrinsic Kirchhoff-Plateau variational problem $\widetilde{F}[C]$, we consider the following examples:

Example 1. Let S denote the Enneper's minimal surface which can be given by the conformal immersion of the plane

$$X(x,y) = \left(x - \frac{x^3}{3} + xy^2, -y + \frac{y^3}{3} - x^2y, x^2 - y^2\right).$$

Let (r,ϑ) be polar coordinates in the plane and consider that Δ_R denotes the domain in the surface S corresponding to the planar disc $x^2 + y^2 \leq R^2$, for any fixed constant $R \neq 0$. Notice that, since the induced metric on S is

$$ds_X^2 = \left(1 + r^2\right)^2\left(dr^2 + r^2 d\vartheta^2\right),$$

then Δ_R is also a geodesic disc in S. Moreover, straightforward calculations give the following quantities for $\partial\Delta_R$,

$$\kappa_g = -\frac{1+3R^2}{R\left(1+R^2\right)^2}\,, \quad \kappa_n = \frac{2\cos(2\vartheta)}{\left(1+R^2\right)^2}\,, \quad \tau_g = \frac{2\sin(2\vartheta)}{\left(1+R^2\right)^2}\,.$$

Then, substituting them in (25), we get after some simplifications

$$\begin{aligned} \alpha\left(1 - 4R^2 + 27R^4\right) &= R^2\left(1+R^2\right)^3 \\ &\times\left(\beta\left[1+3R^2\right] + \sigma R\left[1+R^2\right]^2\right). \end{aligned}$$

Thus, for any fixed $R \neq 0$, one can always choose suitable parameters α, β and $\sigma > 0$ so that above equation is verified and, hence, Δ_R is a critical point of $\widetilde{F}[C]$. For instance, the parameters α and σ can be fixed, while β may be varied. In this case, the length of the boundary curve is changing, while the surface tension $\sigma > 0$ and the coefficient of flexural/torsional rigidity, α, (which are dependent only on the materials of the interface and the rod, respectively) remain invariant.

We point out that Δ_R is not a critical point of the potential energy $E[X]$. Indeed, the geodesic torsion, τ_g, is not constant along $\partial\Delta_R$.

In Figure 2 we show three geodesic discs in the Enneper's minimal surface S, Δ_R, for the values $R = 0.7$, $R = 1.5$ and $R = 2.2$. Fixing $\alpha = \sigma = 1$, above equation tells us that these discs Δ_R are critical for the energy $\widetilde{F}[C]$ for the following values of β, $\beta \simeq 0.4806$, $\beta \simeq -1.1479$ and $\beta \simeq -2.1564$, respectively.

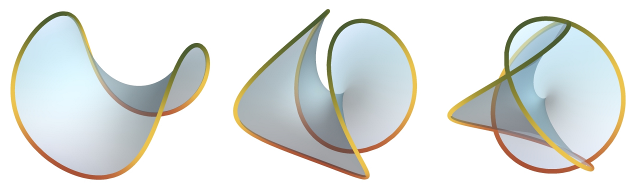

FIG. 2. Three geodesic discs, Δ_R in Enneper's minimal surface and their corresponding boundary curve for the values $R = 0.7$ (Left), $R = 1.5$ (Center) and, $R = 2.2$ (Right).

Example 2. Consider now the catenoid given by the immersion of the plane

$$X(x,y) = \left(R\cosh\left[\frac{x}{R}\right]\cos y, R\cosh\left[\frac{x}{R}\right]\sin y, x\right)$$

where $R > 0$ is a fixed constant. The domain Δ_{01} of the catenoid bounded by two non-geodesic circles of different radi given by $x = x_o$ and $x = x_1$ is a topological annulus. In this setting, we have on $\partial\Delta_{01}$ that

$$\kappa_g = \pm\frac{\sinh(x/R)}{R\cosh^2(x/R)}\,, \quad \kappa_n = \frac{-1}{R\cosh^2(x/R)}\,, \quad \tau_g = 0\,,$$

for each $x = x_o,\ x_1$. Therefore, substitution in the Euler-Lagrange equation (25) gives two equations

$$\left(\alpha - \beta R^2\cosh^2\left[\frac{x}{R}\right]\right)\sinh\left[\frac{x}{R}\right] = \sigma R^3\cosh^4\left[\frac{x}{R}\right]$$

where, again, $x = x_o, x_1$. Then, it is possible to find two of the parameters so that above equations are verified. This allows us to fix one of the parameters, say σ, and leave the ones involving the boundary vary. The right image of Figure 1 shows a domain Δ_{01} in a catenoid.

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Appendix A: Variation of the Twisting Energy

We compute the first variation formula of the twisting energy term appearing in $E[X]$. To do this, we first need to compute the pointwise variation of the geodesic torsion, τ_g. For this purpose, consider a general variation of X whose restriction to the boundary is given by

$$\begin{aligned}(s,t,\varepsilon) &\mapsto C(s,t)+\varepsilon\delta C\\ &= C(s,t)+\varepsilon\left[\varphi(s,t)n+\phi(s,t)T+\psi(s,t)\nu\right].\end{aligned}$$

Now, using that $\delta T = \left[(\delta C)'\right]^{\perp}$ (here, $()^{\perp}$ means orthogonal to T) and $\delta\nu = d\nu\left(\delta C^{T}\right) - \nabla\left(\nu\cdot\delta C\right)$, where $()^{T}$ denotes the tangent component to the immersion X, we obtain the variation of the Darboux frame with respect to δC,

$$\begin{cases} n_\varepsilon = \left(\tau_g\psi-\kappa_g\phi-\varphi'\right)T+\left(\partial_n\psi+\tau_g\phi+[2H-\kappa_n]\varphi\right)\nu\,,\\ T_\varepsilon = \left(\varphi'+\kappa_g\phi-\tau_g\psi\right)n+\left(\psi'+\kappa_n\phi+\tau_g\varphi\right)\nu\,,\\ \nu_\varepsilon = \left([\kappa_n-2H]\varphi-\tau_g\phi-\partial_n\psi\right)n+\left(-\tau_g\varphi-\kappa_n\phi-\psi'\right)T, \end{cases}$$

where ∂_n represents the derivative in the conormal direction. Moreover, we have that

$$\tau_g = -\nu'\cdot n = -\frac{\nu_\eta}{\|C_\eta\|}\cdot n$$

where η denotes an arbitrary parameter. Therefore, using the variation of Darboux frame and after long straightforward computations differentiating above relation with respect to δC, we obtain

$$\begin{aligned}\delta\tau_g &= \left(\partial_n\left[\nu\cdot\delta C\right]+\left[2H-\kappa_n\right]n\cdot\delta C\right)'+\left(\tau_g T\right)'\cdot\delta C\\ &\quad -\left(T\times T'\right)\cdot\left(\delta C\right)'.\end{aligned}$$

Hence, we use it together with integration by parts to obtain

$$\begin{aligned}&\delta\left(\int_C \tau_g^2\,ds\right) = \int_C\left(2\tau_g\delta\left[\tau_g\right]ds+\tau_g^2\delta\left[ds\right]\right)\\ &= \int_C\left(2\tau_g'\left[\kappa_n-2H\right]n+\tau_g^2T'+2\left[\tau_g T\times T'\right]'\right)\cdot\delta C\,ds\\ &-2\int_C\tau_g'\partial_n\psi\,ds+\left(2\tau_g\partial_n\psi-\left[2\tau_g\left(\kappa_n-2H\right)n+2\tau_g T\times T'\right.\right.\\ &\left.\left.-\tau_g^2T\right]\cdot\delta C\right|_{\partial C}.\end{aligned}$$

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