Chapter 1

Motion: An Introduction

1.1 Overview

This notes are designed as a gentle introduction to the use of Clifford algebras in robot kinematics. In the process, other key concepts will be reviewed, concepts that Clifford algebras build upon. Basic group theory, linear spaces, Grassman spaces and Lie algebras, as well as line geometry and projective geometry, will be introduced. Each of these topics is accompanied by examples and applications of increasing difficulty.

1.2 Introduction

Kinematics is defined as the study of the motion, regardless of the forces causing it and caused by it. Motion is a concept that includes position and its derivatives, mainly velocity and acceleration.

The subjects that we study are modeled as rigid bodies. A rigid body is a set of particles such that the distance between them remains fixed. This means that, unlike individual particles, we can define not only location, velocity and acceleration of a particle in the body, but also orientation, angular velocity and angular acceleration of the body. Orientation is defined as a shortcut to avoid describing the motion of a set of particles in the body; for completely defining the position of a rigid body, we need to define the location of at least three particles on it, $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$, that is 9 parameters. However, the three locations are not independent, because the distance between each two particles must remain constant; this adds three constraints to the set of nine parameters. Or we can define the location of one particle and the orientation of a frame attached to the body at that point, in which case we just need to define six parameters, given by $\mathbf{d}$ and $[R]$ in Fig 1.1. The concept of orientation is also well suited for algebraic operations.

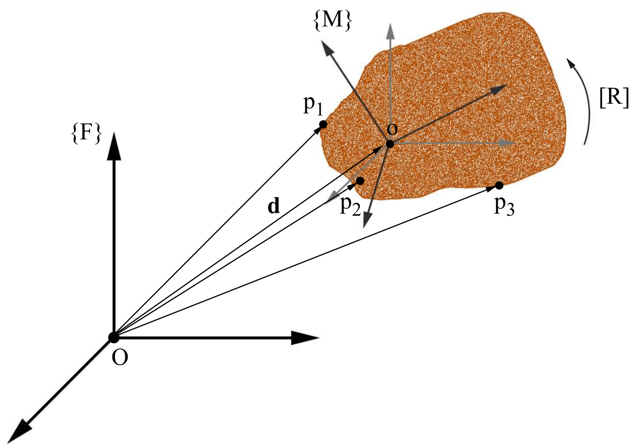

Figure 1.1: Description of the motion of a rigid body

A transformation that preserves the distance between points is called an *isometry* and motion is one of them. There is no fundamental difference in describing a translation (a change in location) and a rotation (a change in orientation) at the level of an individual particle. When the particles form a rigid body, we can characterize the translation because all the particles perform the same displacement, while during a rotation the particles of the rigid body move in different trajectories (still keeping the distance among them), and there is always a set of particles that do not move.

To model the motion of a rigid body, we consider the whole space as moving attached to it; this allows us to use some results in group theory. Figure 1.2 shows a rigid body undergoing a translation and a rotation.

In this chapter, we will become familiar with the basic concepts of motion and the most common tools to describe it.

1.3 The finite displacement

The motion of a rigid body is modeled as the motion of a three-dimensional space. We attach a coordinate frame to the rigid body, in which the coordinates of points of the body remain fixed (Figure 1.3).

Every point can be given coordinates with respect to a frame; we use right-handed orthogonal frames. In Figure 1.3, a point **P** has coordinates (X_1, X_2, X_3) when measured in frame $\{F\}$ and $\mathbf{p} = (x_1, x_2, x_3)$ with respect to frame $\{M\}$.

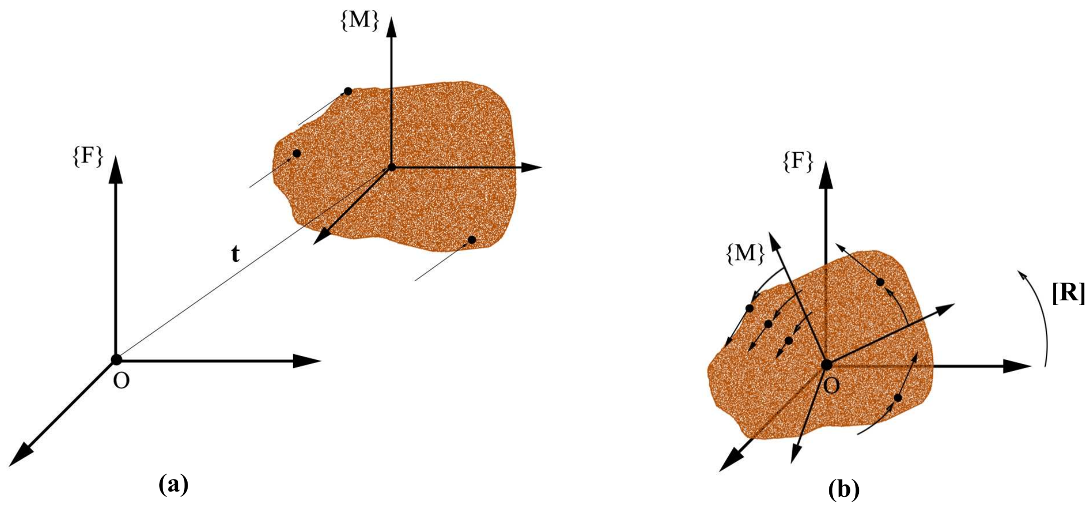

Figure 1.2: (a) A pure translation and (b) a pure rotation of a rigid body

A finite motion is called a *displacement*. A displacement can be either a translation, a rotation, or a combination of both. We will see how a general motion can always be seen as a composition of a rotation and a translation.

1.3.1 Translations

During a translation, all points of the space move by the same amount in the same direction. If the direction and magnitude of translation is given by a non-unitary vector $\mathbf{t}$ as in Figure 1.2 (a), a point that had coordinates $\mathbf{x}$ in frame $\{F\}$ before the translation, is transformed to a point of coordinates

$$\mathbf{X} = \mathbf{x} + \mathbf{t} \tag{1.1}$$

in the fixed frame $\{F\}$ after the translation.

Here, both points and vectors have coordinates in the three-dimensional space; however, they are different objects. The same vector $\mathbf{t}$ is used to transform any point of the space to the new coordinates after the translation. It is a free vector whose origin can be located anywhere in the 3-D space.

Notice that translations are not linear transformations. If we denote the translation as $g_t(\mathbf{x}) = \mathbf{x}+\mathbf{t}$,

$$g_t(\mathbf{x} + \mathbf{y}) = \mathbf{x} + \mathbf{y} + \mathbf{t} \neq g_t(\mathbf{x}) + g_t(\mathbf{y}) = \mathbf{x} + \mathbf{y} + 2\mathbf{t} \tag{1.2}$$

Let us now show that translations are isometries, that is, they keep the distance between points.

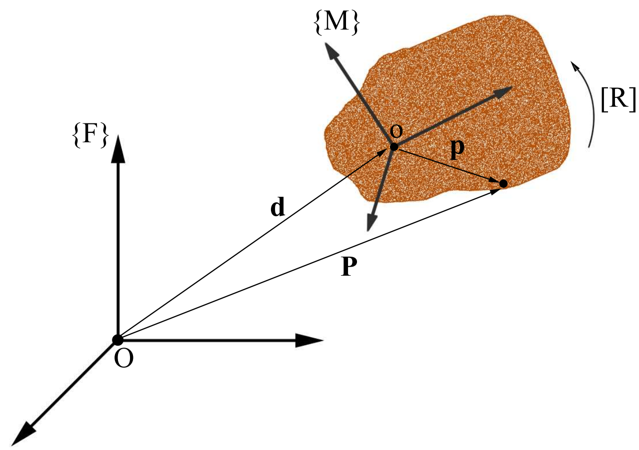

Figure 1.3: A rigid displacement

Consider two points $\mathbf{X}$ and $\mathbf{Y}$, $\mathbf{X} = \mathbf{x} + \mathbf{t}$ and $\mathbf{Y} = \mathbf{y} + \mathbf{t}$, before and after a translation by a vector $\mathbf{t}$. The distance between them is

$$d^2 = (\mathbf{X} - \mathbf{Y}) \cdot (\mathbf{X} - \mathbf{Y}) = (\mathbf{x} + \mathbf{t} - (\mathbf{y} + \mathbf{t})) \cdot (\mathbf{x} + \mathbf{t} - (\mathbf{y} - \mathbf{t})) = (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}). \tag{1.3}$$

Last, let us notice that translations do not have invariants, that is, do not leave any point fixed. For a point to be fixed by a translation g_t, there must exist a point $\mathbf{x}$ such that $g_t(\mathbf{x}) = \mathbf{x} + \mathbf{t} = \mathbf{x}$. But this is true only if $\mathbf{t} = \vec{0}$, which is the zero translation and leaves all points in the space fixed.

1.3.2 Rotations

During a rotation, a subspace of points (in the body or outside) can be found that remains fixed, while the rest of the points of the space move by different amounts and in different directions according to their location with respect to the fixed points. Because of the distance-preserving property of the rigid body, it is enough to define the change of direction along three independent directions to define a rotation. We attach a *moving* frame $\{M\}$ to a point of the rigid body, see Figure 1.2 (b). The *orientation* of the body is defined as the coordinates of the three perpendicular vectors that form the moving frame, expressed in the fixed frame.

The most common way of representing rotations is by using matrices to represent rotations. The matrix that contains the expression of the column vectors of the moving frame $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ expressed in the fixed frame,

$$[R] = \begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \end{bmatrix}, \tag{1.4}$$

defines the rotation from an initial position in which the fixed and moving frames were coincident, to the final position of the moving frame. At the same time, the matrix $[R]$ defines also the change in coordinates from points of the body expressed in coordinates of the moving frame, to the expression of the same point in the fixed frame, see Figure 1.3. A point that had coordinates $\mathbf{x}$ (expressed in the fixed frame) before the rotation, is transformed to a point of coordinates

$$\mathbf{X} = [R]\mathbf{x} \tag{1.5}$$

after the rotation.

Rotations are linear transformations. If we denote the transformation as $r(\mathbf{x}) = [R]\mathbf{x}$,

$$r(\mathbf{x}+\mathbf{y}) = [R](\mathbf{x}+\mathbf{y}) = [R]\mathbf{x} + [R]\mathbf{y} = r(\mathbf{x}) + r(\mathbf{y}). \tag{1.6}$$

Because we are using orthonormal frames, the rotation matrix is an orthogonal matrix, that is, its columns are mutually orthogonal and unitary. This, together with the fact that we use right-handed frames, leads to the result:

Proposition 1.3.1 *A rotation matrix satisfies the following properties:*

- $[R][R]^T = [R]^T[R] = [I]$
- $\det[R] = +1$

Proof: Exercise.

Let us check now that rotations are isometries. Consider $\mathbf{X} = [R]\mathbf{x}$ and $\mathbf{Y} = [R]\mathbf{y}$ being two points transformed by the same rotation. The distance between them is

$$\begin{aligned}(\mathbf{X}-\mathbf{Y})\cdot(\mathbf{X}-\mathbf{Y}) =&([R]\mathbf{x}-[R]\mathbf{y})\cdot([R]\mathbf{x}-[R]\mathbf{y}) = ([R](\mathbf{x}-\mathbf{y}))\cdot([R](\mathbf{x}-\mathbf{y})) = \\ &([R](\mathbf{x}-\mathbf{y}))^T([R](\mathbf{x}-\mathbf{y})) = (\mathbf{x}-\mathbf{y})^T[R]^T[R](\mathbf{x}-\mathbf{y}) = (\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y})\end{aligned} \tag{1.7}$$

We can use our knowledge of matrix algebra to determine whether there are points that remain fixed during a rotation (they are called *invariants*) and how to find these points.

Let us look at the eigenvalues of $[R]$. A scalar λ is an eigenvalue of a matrix $[R]$ if $[R]\mathbf{x} = \lambda\mathbf{x}$. To obtain some information about their values, we can compute the length of the eigenvector before and after the transformation. For doing that, we compute the dot product; if we transpose the above expression (taking into account that λ may be a complex number), and we multiply them together, we obtain

$$\mathbf{x}^T[R]^T[R]\mathbf{x} = \bar{\lambda}\lambda\mathbf{x}^T\mathbf{x}, \tag{1.8}$$

and using the fact that $[R]^T[R] = [I]$ and $\bar{\lambda}\lambda = |\lambda|^2$,

$$|\lambda|^2 = 1 \tag{1.9}$$

for all λ being an eigenvalue of the rotation matrix. This, together with the fact that $\det[R] = \lambda_1\lambda_2\lambda_3 = +1$, leaves us with the following eigenvalues:

Proposition 1.3.2 *A three-dimensional rotation matrix has the following eigenvalues:*

- $\lambda_1 = 1$
- $\lambda_2 = e^{i\theta}$
- $\lambda_3 = e^{-i\theta}$

Proof: See the construction above. For a different construction involving the characteristic polynomial of the rotation matrix, see [6].

A fixed point will have the same coordinates before and after the transformation, that is,

$$[R]\mathbf{x} = \mathbf{x}. \tag{1.10}$$

From here we have

$$([R] - [I])\mathbf{x} = \vec{0}. \tag{1.11}$$

The obvious solution is $\mathbf{x} = \vec{0}$, which tells us that the origin is an invariant. A possible subspace of fixed points exists if $\det([R] - [I]) = 0$. This is always true, as we know that one of the eigenvalues of $[R]$ is $\lambda_1 = 1$. We will find not a point, but a whole direction of points being invariants. The only other possibility of fixed points is when the other two eigenvalues also become equal to 1; in that case we get the identity matrix. We know that the direction of fixed points includes the origin; it is called the *rotation axis* and it can be found using Cayley's formula. For the derivation of this formula we follow [7].

We start with the length-preserving property of rotations as seen in Eq.(1.8). We can express it as $\mathbf{X} \cdot \mathbf{X} - \mathbf{x} \cdot \mathbf{x} = 0$. This is transformed to

$$(\mathbf{X} - \mathbf{x}) \cdot (\mathbf{X} + \mathbf{x}) = 0, \tag{1.12}$$

so that the addition and difference vectors are perpendicular. Notice that, if we express $\mathbf{X} = [R]\mathbf{x}$, we obtain the expression,

$$\mathbf{X} - \mathbf{x} = [R - I][R + I]^{-1}(\mathbf{X} + \mathbf{x}). \tag{1.13}$$

We define the matrix

$$[B] = [R - I][R + I]^{-1}. \tag{1.14}$$

By construction we have seen that applying $[B]$ to a vector $\mathbf{y}$ makes it perpendicular. Hence, $\mathbf{y}^T[B]\mathbf{y} = 0$. From this equation we can see that $[B]$ is a skew-symmetric matrix, that is, $[B]^T =$

$-[B]$ and its elements are

$$[B] = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}. \quad (1.15)$$

The only three different elements of the matrix can be assembled in the vector $\mathbf{b} = (b_1, b_2, b_3)$, and the matrix product with $[B]$ can be substituted by the vector cross product, $[B]\mathbf{y} = \mathbf{b} \times \mathbf{y}$.

From Eq.(1.14) we can derive the expression of $[R]$ as a function of $[B]$, also known as Cayley's formula,

$$[R] = [I - B]^{-1}[I + B]. \quad (1.16)$$

When we substitute this expression in Eq.(1.11), we obtain

$$[B]\mathbf{x} = 0, \quad (1.17)$$

hence $\mathbf{b} \times \mathbf{x} = 0$, which means that $\mathbf{b}$ is a solution of the fixed point equation. The rotation axis is given by the direction of vector $\mathbf{b}$.

The set of invariant points that form the direction $\mathbf{b}$ defines the rotation axis and it is also called Rodrigues' vector. It can also be shown that the length of this vector is $|\mathbf{b}| = \tan\frac{\phi}{2}$, where ϕ is the rotation angle about the rotation axis.

1.3.3 General finite displacements

A general finite displacement can be always described as a composition of a translation and a rotation. In this context, composition means just the concatenation of two transformations, one realized after the other. We will formalize this within the context of group theory.

We will write the general displacement g_T as composed of a rotation by an orthogonal matrix $[R]$ and a translation by a vector $\mathbf{d}$,

$$\mathbf{X} = g_T(\mathbf{x}) = [R]\mathbf{x} + \mathbf{d}. \quad (1.18)$$

It is easy to see that the properties of rotation matrices and translation vectors studied above hold for the general displacement. We will study the invariants of a general displacement later; we are now going to derive some formulas that will be useful for us when dealing with displacements.

Composition of displacements

Let us derive the displacement that we obtain when we apply two successive displacements to a point. Let g_{T1} transform the point $\mathbf{X} = [R_1]\mathbf{y} + \mathbf{d}_1$ and g_{T2} be such that $\mathbf{y} = [R_2]\mathbf{x} + \mathbf{d}_2$, see Figure 1.4. We obtain that the total transformation is

$$\mathbf{X} = g_{T3}(\mathbf{x}) = g_{T1}(\mathbf{y})g_{T2}(\mathbf{x}) = [R_1][R_2]\mathbf{x} + ([R_1]\mathbf{d}_2 + \mathbf{d}_1) \quad (1.19)$$

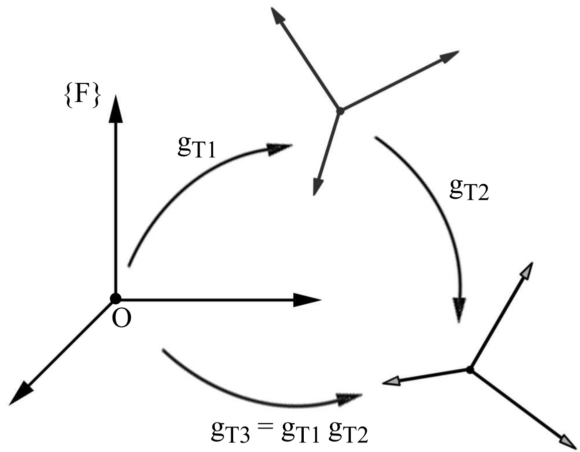

Figure 1.4: Composition of displacements

Inverse displacement

Given a displacement $g_T = ([R], \mathbf{d})$, we want now to calculate a displacement that, when composed with it, brings all the points of the space to their initial positions. For doing that, just consider the composition of displacements in Eq.(1.19) and solve for $[R_2]$ and $\mathbf{d}_2$ such that the resulting matrix is the identity and the resulting translation vector is zero. We obtain

$$g_{T^{-1}} = ([R]^T, -[R]^T\mathbf{d}) \tag{1.20}$$

Relative displacements

A relative displacement expresses a displacement from an initial point to a final point, *all measured in the fixed reference frame*. Let $\mathbf{x}$ be a point expressed in the moving frame, and let g_{T1} and g_{T2} be two different displacements of this point with respect to a fixed frame. We want to calculate what is the displacement between the first and the second motion, relative to the fixed frame. That is, from the coordinates $\mathbf{X}_1 = g_{T1}(\mathbf{x})$ and $\mathbf{X}_2 = g_{T2}(\mathbf{x})$, we want to obtain the relative transformation $\mathbf{X}_2 = g_{T12}(\mathbf{X}_1)$ as shown in Figure 1.5.

Substituting the value of $\mathbf{x}$ from the first equation in the second, we obtain

$$\mathbf{X}_2 = g_{T12}(\mathbf{X}_1) = [R_2][R_1]^T\mathbf{X}_1 + (\mathbf{d}_2 - [R_2][R_1]^T\mathbf{d}_1), \tag{1.21}$$

and in general, the relative displacement is

$$\mathbf{X}_2 = g_{T12}(\mathbf{X}_1) = (g_{T2}g_{T1}^{-1})(\mathbf{X}_1) \tag{1.22}$$

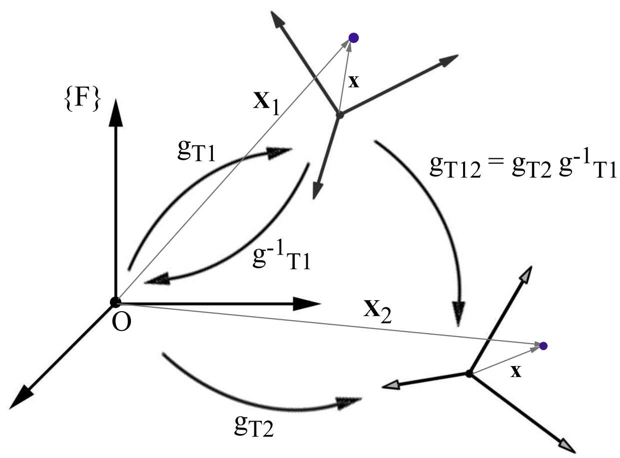

Figure 1.5: A relative displacement

1.3.4 The invariants of a general displacement

We may be interested now in studying the invariants of a general displacement, that is, whether there is a point or set of points such that $g_T(\mathbf{x}) = \mathbf{x}$. This gives the expression

$$[R - I]\mathbf{x} = -\mathbf{d}, \tag{1.23}$$

and knowing that $\lambda = 1$ is an eigenvalue of the rotation matrix and hence $[R - I]$ is singular, we can determine that there are no fixed points in a general displacement. However, there is a *line* which is an invariant of the displacement. In order to calculate this line, we need to study some line geometry. We will come back to this calculation after next section.

1.4 A little bit of line geometry

This section is intended as an introduction to become familiar with line geometry. At this point, we are interested in having a good, useful definition of a line in space. Later on we may be interested in how to define planes, volumes, etc., and we will see that we can do it in a similar fashion.

Lines are heavily used in spatial kinematics, and the reason was pointed out at the end of last section: for every displacement, we can find an invariant line. We find these invariant lines everywhere in mechanisms that perform motion; they usually define where the joints are located.

A line has a direction and it has a location in space. The parameterized expression of a line uses a point $\mathbf{c}$ on the line, a direction vector $\mathbf{s}$ and a parameter t, describing the points belonging to the line as

$$\mathcal{L}: \quad \mathbf{c} + t\mathbf{s}, \quad t \in \mathbb{R} \tag{1.24}$$

However, this description is not very good for algebraic operations. We want to determine distances between lines, angles between lines and so on. We use a description of a line called the *Plücker coordinates*. They define the line as two vectors, a *direction* vector and a *moment* vector,

$$\mathsf{L} = \mathbf{s} + \epsilon\mathbf{s}^0 = \mathbf{s} + \epsilon\mathbf{c} \times \mathbf{s}, \tag{1.25}$$

where $\mathbf{s}$ is a three-dimensional vector indicating the direction of the line and $\mathbf{s}^0$ is also a three-dimensional vector, called the moment of the line, and it is obtained by computing the cross product of any point $\mathbf{c}$ on the line times the direction $\mathbf{s}$. The symbol ϵ is the *dual unit* and it is used to separate both vectors; it has the property that $\epsilon^2 = 0$. Figure 1.6 shows the parameters that define a line.

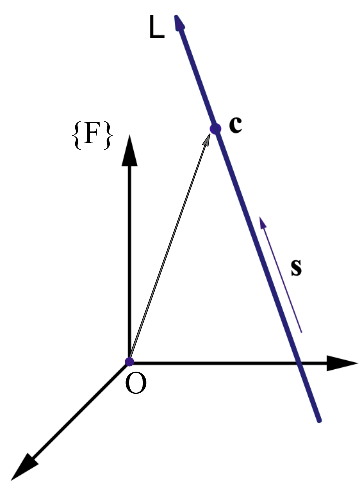

Figure 1.6: A line defined using Plücker coordinates

In a more general context, an object of the form $\mathsf{S} = \mathbf{v} + \epsilon\mathbf{w}$ is known as a *dual vector* or a *screw*, and lines are particular cases of dual vectors, in which the following constraints hold:

$$|\mathbf{v}| = 1 \quad \text{and} \quad \mathbf{v} \cdot \mathbf{w} = 0. \tag{1.26}$$

Vector algebra extends naturally to dual vector algebra by operating component-wise and using the property of the dual unit. (We can similarly define *dual numbers* as pairs of scalars separated by the dual unit).

1.4.1 Dual vector algebra

We can naturally extend to dual vectors the operations of three-dimensional vector algebra. Addition of dual vectors is performed component-wise. We can also extend both the dot product and the cross product of vectors. Let $\mathsf{S}_1 = \mathbf{v}_1 + \epsilon\mathbf{w}_1$ and $\mathsf{S}_2 = \mathbf{v}_2 + \epsilon\mathbf{w}_2$ be two dual vectors. The *dual* *dot product* is defined as:

$$\mathsf{S}_1 \cdot \mathsf{S}_2 = \mathbf{v}_1 \cdot \mathbf{v}_2 + \epsilon(\mathbf{v}_1 \cdot \mathbf{w}_2 + \mathbf{w}_1 \cdot \mathbf{v}_2), \tag{1.27}$$

where $\cdot$ is the usual dot product. Similarly, we define the *dual cross product* of two dual vectors as

$$\mathsf{S}_1 \times \mathsf{S}_2 = \mathbf{v}_1 \times \mathbf{v}_2 + \epsilon(\mathbf{v}_1 \times \mathbf{w}_2 + \mathbf{w}_1 \times \mathbf{v}_2), \tag{1.28}$$

where $\times$ is the usual cross product of three-dimensional vectors.

The dual dot and cross product have geometric meaning; it can be derived from the geometry of vector algebra. We are going to show this meaning for the case in which the dual vectors are lines.

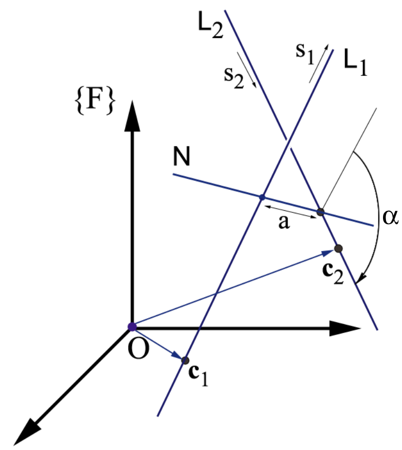

Figure 1.7: Dual vector geometry

Let $\mathsf{L}_1 = \mathbf{s}_1 + \epsilon \mathbf{c}_1 \times \mathbf{s}_1$ and $\mathsf{L}_2 = \mathbf{s}_2 + \epsilon \mathbf{c}_2 \times \mathbf{s}_2$ be two lines arbitrarily located in space as shown in Figure 1.7. The dual number obtained as the dual dot product of the lines is

$$\mathsf{L}_1 \cdot \mathsf{L}_2 = \cos\alpha - \epsilon a \sin\alpha, \tag{1.29}$$

where α is the angle between the direction vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ and a is the distance between the two lines along their common normal line. Similarly, the dual vector obtained by the cross product of the lines is

$$\mathsf{L}_1 \times \mathsf{L}_2 = (\sin\alpha + \epsilon a \cos\alpha)\mathsf{N}, \tag{1.30}$$

where N is the common normal line between L_1 and L_2.

Exercise: Prove Eqs.(**??**). Use points $\mathbf{c}_1$, $\mathbf{c}_2$ as the intersection points of L_1, L_2 with the common normal line.

What happens if the dual vectors L_1 and L_2 are not lines? Remember that the only conditions for a dual vector to be a line are given in Eq.(1.26). If the real part (the direction) of the dual vector is not unit, we can compute its length, which we will denote as m. If the dual part (the moment) is not perpendicular to the real part, we can always write it as a perpendicular component plus a component parallel to the direction. So finally we can express a general dual vector, also called a *screw*, as

$$\mathsf{S} = m\mathbf{s} + \epsilon(m\mathbf{c} \times \mathbf{s} + mk\mathbf{s}), \tag{1.31}$$

where the magnitude k along the real part is called the *pitch* of the screw.

1.4.2 More line geometry

Sometimes we may be interested in computing a point in the line, given its Plücker coordinates. This can be done in general, for a screw $\mathsf{S} = \mathbf{v} + \epsilon\mathbf{w}$,

$$\mathbf{c} = \frac{\mathbf{v} \times \mathbf{w}}{\mathbf{v} \cdot \mathbf{v}}. \tag{1.32}$$

We can see how it works: take the general expression of a screw in Eq.(1.31), and let us compute the cross product of $\mathbf{w} = m\mathbf{c} \times \mathbf{s} + mk\mathbf{s}$ by $\mathbf{v} = m\mathbf{s}$,

$$m\mathbf{s} \times (m\mathbf{c} \times \mathbf{s} + mk\mathbf{s}) = m^2\mathbf{s} \times (\mathbf{c} \times \mathbf{s}) + m^2k\mathbf{s} \times \mathbf{s}. \tag{1.33}$$

Now we apply the identity $a \times (b \times c) = ba \cdot c - ca \cdot b$ to obtain

$$m^2\mathbf{s} \times (\mathbf{c} \times \mathbf{s}) = m^2(\mathbf{c}(\mathbf{s} \cdot \mathbf{s}) - \mathbf{s}(\mathbf{s} \cdot \mathbf{c})) = m^2\mathbf{c} \tag{1.34}$$

if we consider $\mathbf{c}$ as the point perpendicular to the direction $\mathbf{s}$ - remember that we can take any point on the line. Notice also that m^2 is the dot product of the direction of the dual vector by itself.

The fact that any point can be used to compute the moment of the line can be easily seen: imagine that $\mathbf{c}$ is the point on the line perpendicular to the direction, and consider now the point $\mathbf{k} = \mathbf{c} + t\mathbf{s}$ which has a component along the direction $\mathbf{s}$. Compute the moment of the line using $\mathbf{k}$,

$$\mathbf{k} \times \mathbf{s} = (\mathbf{c} + t\mathbf{s}) \times \mathbf{s} = \mathbf{c} \times \mathbf{s} + t\mathbf{s} \times \mathbf{s} = \mathbf{c} \times \mathbf{s}. \tag{1.35}$$

The component along the direction does not contribute to the moment.

From all the information on this section we can define how to compute the common normal line, the line which is perpendicular to two given lines. Using Eq.(1.30), we obtain a screw which is perpendicular to both lines. In order to make it a line,

- Make the direction a unit vector, $\mathbf{n}$.
- Find a point $\mathbf{p}$ in the screw as indicated above.
- Construct the common normal line as $\mathsf{N} = \mathbf{n} + \epsilon\mathbf{p} \times \mathbf{n}$.

1.4.3 Line motion

A question that relates to our interests is, how do we specify the motion of a line? We need to be able to apply a displacement to a line and obtain the displaced line. One way is to use the rotation matrix to compute the change in orientation of the direction, and apply separately the displacement to a point on the line, assembling the Plücker coordinates of the line afterwards. We can also create a matrix that captures all these operations.

Let $[R]$ be the rotation matrix and $\mathbf{d} = (d_1, d_2, d_3)$ the translation vector of a spatial displacement. We can create the *dual orthogonal matrix* $[\hat{T}]$ as

$$[\hat{T}] = [R] + \epsilon[R]^0 = [R] + \epsilon[D][R], \tag{1.36}$$

with

$$[D] = \begin{bmatrix} 0 & -d_3 & d_2 \\ d_3 & 0 & -d_1 \\ -d_2 & d_1 & 0 \end{bmatrix} \tag{1.37}$$

which, as any skew-symmetric matrix, has the property $[D]\mathbf{y} = \mathbf{d} \times \mathbf{y}$. We operate dual vectors and dual matrices distributing over the two components and applying the dual unit property $\epsilon^2 = 0$, to get

$$\mathbf{S} + \epsilon\mathbf{C} \times \mathbf{S} = ([R] + \epsilon[R]^0)(\mathbf{s} + \epsilon\mathbf{c} \times \mathbf{s}) = [R]\mathbf{s} + \epsilon\big(([R]\mathbf{c} + \mathbf{d}) \times [R]\mathbf{s}\big), \tag{1.38}$$

if we notice that $[R](\mathbf{c} \times \mathbf{s}) = [R]\mathbf{c} \times [R]\mathbf{s}$ when $[R]$ is an orthogonal matrix.

1.5 The screw axis of a general displacement and Chasles' Theorem

We now go back to finding the invariants of a general displacement. Based on Eq.(1.23), we decided that there are no fixed points. We know that for every rotation there is a fixed direction, called the rotation axis. Is there any other entity that remains fixed by the action of a general displacement? The answer is yes, there is an invariant line, called the *screw axis of the displacement*.

To find this line, let us apply a general displacement to a line, using the dual orthogonal matrix. A fixed line $\mathsf{S} = \mathbf{s} + \epsilon\mathbf{c} \times \mathbf{s}$ is such that

$$[\hat{T}]\mathsf{S} = \mathsf{S}. \tag{1.39}$$

We can see the effect of this transformation in the real and dual parts of the line. For the real part,

$$[R]\mathbf{s} = \mathbf{s}, \tag{1.40}$$

and we saw that this equation always have a vector as a solution, the vector $\mathbf{b}$ defining the rotation axis, constructed from the skew-symmetric matrix $[B]$. For the dual part,

$$[R](\mathbf{c} \times \mathbf{s}) + [D][R]\mathbf{s} = \mathbf{c} \times \mathbf{s}. \tag{1.41}$$

If we consider $\mathbf{s} = \mathbf{b}/|\mathbf{b}|$, the rotation axis from the previous equation, then $[R]\mathbf{s} = \mathbf{s}$ and we can collect as

$$([R]\mathbf{c} + \mathbf{d}) \times \mathbf{s} = \mathbf{c} \times \mathbf{s}. \tag{1.42}$$

The cross product does not get affected by the addition of a component along the direction $\mathbf{s}$, so we add a component $t\mathbf{s}$ to the transformed point to get the equation

$$[R]\mathbf{c} + \mathbf{d} + t\mathbf{s} = \mathbf{c}, \tag{1.43}$$

that we want to solve for the point $\mathbf{c}$. Rearranging the terms and using Cayley's equation for $[R]$,

$$2[B]\mathbf{c} = -[I + B](\mathbf{d} + t\mathbf{s}), \tag{1.44}$$

and using the skew-symmetry of $[B]$,

$$2\mathbf{b} \times \mathbf{c} = -\mathbf{d} - t\mathbf{s} - \mathbf{b} \times \mathbf{d}. \tag{1.45}$$

To solve for $\mathbf{c}$, we compute the cross product with $\mathbf{b}$,

$$2\mathbf{b} \times (\mathbf{b} \times \mathbf{c}) = -\mathbf{b} \times \mathbf{d} - \mathbf{b} \times (\mathbf{b} \times \mathbf{d}), \tag{1.46}$$

which simplifies to

$$\mathbf{c} = \frac{1}{2}\Big(\frac{\mathbf{b} \times \mathbf{d}}{\mathbf{b} \cdot \mathbf{b}} + \big(\frac{\mathbf{b} \cdot \mathbf{d}}{\mathbf{b} \cdot \mathbf{b}}\mathbf{b} - \mathbf{d}\big)\Big). \tag{1.47}$$

Recall that this point will not be fixed by the transformation; it is just one of the points lying on the line so that the moment of the line does not change due to the displacement. Just how much does this particular point (which is, BTW, the point perpendicular to the direction $\mathbf{s}$) translate along the line?

In general, the translation vector $\mathbf{d}$ of the displacement will not be perpendicular to the direction of the rotation axis. The perpendicular component is given by the expression in the second parenthesis in Eq.(1.47). The component parallel to the rotation axis is given by its projection and will define the amount of slide along the line. We can write

$$t = \frac{\mathbf{b} \cdot \mathbf{d}}{|\mathbf{b}|}. \tag{1.48}$$

Chasles' theorem can be used to summarize the results of this section. Figure 1.8 shows the equivalent descriptions of a displacement.

Theorem 1.5.1 *(Chasles, approx. 1830). A general displacements in 3-dimensional space is equivalent to a screw motion consisting of a rotation of angle ϕ about and a translation t along a line $\mathsf{S} = \mathbf{s} + \epsilon\mathbf{s}^0$.*

The direction $\mathbf{s}$ and rotation angle ϕ are computed from the rotation. The moment vector $\mathbf{s}^0$ and the slide are computed using the equations just derived above. In total, we need to define six parameters: four to define the direction and location of the line, and two to define the rotation and slide values. The screw axis is a very efficient way of representing a general displacement.

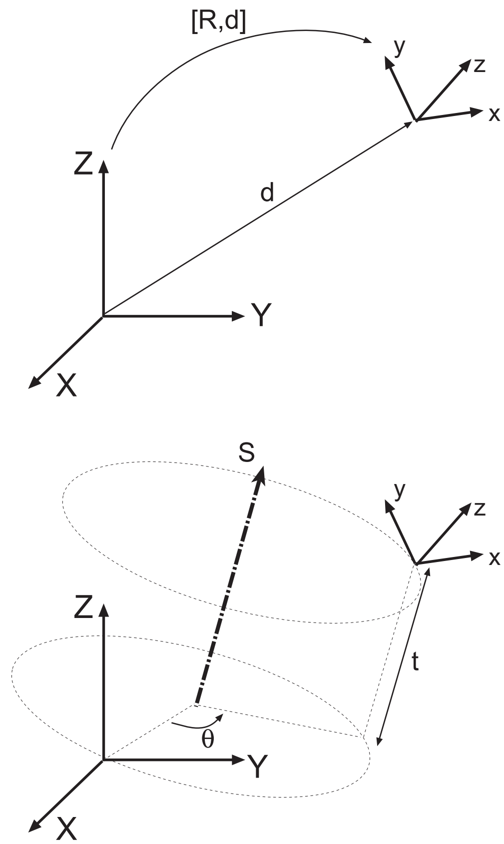

Figure 1.8: The screw axis, rotation angle and slide of a displacement

1.6 More on Matrix Representation

1.6.1 How to create a rotation matrix

When we covered rotations, we defined the rotation matrix as containing as columns the coordinates of the vectors of the moving frame, expressed in the fixed frame. This description, even though it is correct, is of little practical use when we want to compute the matrix defining a rotation.

Methods to generate a rotation matrix include: Euler angles, roll-pitch-yaw angles, longitude-latitude-roll angles, product of two reflections, use of the rotation axis and rotation angle. We will cover here Euler angles and the use of the rotation axis; for more information, see [5] or [3].

Euler angles

A rotation about one of the coordinate axes of the fixed frame adopts a very simple expression and it is also easy to visualize. The coordinate rotations about axes X, Y and Z are given by the matrices

$$[X(\alpha)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \quad [Y(\beta)] = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \quad [Z(\theta)] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.49}$$

We can create a general rotation as the composition of three rotations about perpendicular axes. We could use $\mathsf{X}-\mathsf{Y}-\mathsf{Z}$,

$$[R(\alpha,\beta,\theta)] = [X(\alpha)][Y(\beta)][Z(\theta)], \tag{1.50}$$

or any other combination, like $\mathsf{Z}-\mathsf{Y}-\mathsf{Z}$. Notice that the first rotation happens about X, and the second rotation is about the *rotated* Y axis, not the original one, and the third rotation is about an axis Z that has been transformed by the two previous rotations; see Figure 1.9.

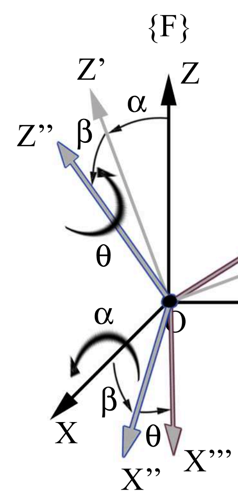

Figure 1.9: Euler angles

A somehow similar way of describing a rotation is by using three coordinate rotations about *fixed* axis. See [5] for a description of this method.

Equivalent axis-angle representation and Euler parameters

We proved in section 1.3.2 that a rotation can be characterized by the rotation axis and rotation angle about the axis. Using Cayley's equation, we can express the components of the rotation matrix as a function of the parameters of the vector $\mathbf{b}$. Recall that $\mathbf{b} = (b_1, b_2, b_3) = \tan\frac{\phi}{2}(s_x, s_y, s_z)$. We obtain

$$[R] = [I-B]^{-1}[I+B] = \frac{1}{1+b_1^2+b_2^2+b_3^2}\begin{bmatrix} 1+b_1^2-b_2^2-b_3^2 & 2(b_1b_2-b_3) & 2(b_1b_3+b_2) \\ 2(b_1b_2+b_3) & 1-b_1^2+b_2^2-b_3^2 & 2(b_2b_3-b_1) \\ 2(b_1b_3-b_2) & 2(b_2b_3+b_1) & 1-b_1^2-b_2^2+b_3^2 \end{bmatrix} \tag{1.51}$$

We can see that using the invariant axis and angle to define the rotation matrix gives a not so simple expression. There is a similar expression of the rotation matrix as a function of the rotation axis and angle. If we multiply vector $\mathbf{b}$ by $\cos\frac{\phi}{2}$, with $\cos\frac{\phi}{2} \neq 0$, we can define the following four coefficients, which are called the Euler parameters of the rotation,

$$\begin{aligned} c_1 &= \cos\frac{\phi}{2} b_1 = \sin\frac{\phi}{2} s_1, \\ c_2 &= \cos\frac{\phi}{2} b_2 = \sin\frac{\phi}{2} s_2, \\ c_3 &= \cos\frac{\phi}{2} b_3 = \sin\frac{\phi}{2} s_3, \\ c_0 &= \cos\frac{\phi}{2}. \end{aligned} \tag{1.52}$$

The rotation matrix can be written then as

$$[R] = \frac{1}{c_0^2 + c_1^2 + c_2^2 + c_3^2} \begin{bmatrix} c_0^2 + c_1^2 - c_2^2 - c_3^2 & 2(c_1c_2 - c_0c_3) & 2(c_1c_3 + c_0c_2) \\ 2(c_1c_2 + c_0c_3) & c_0^2 - c_1^2 + c_2^2 - c_3^2 & 2(c_2c_3 - c_0c_1) \\ 2(c_1c_3 - c_0c_2) & 2(c_2c_3 + c_0c_1) & c_0^2 - c_1^2 - c_2^2 + c_3^2 \end{bmatrix} \tag{1.53}$$

This parameters, which do not seem to add any advantage in the description of the rotation, will be revisited when we study Clifford algebras.

1.6.2 Homogeneous matrix representation

The homogeneous matrix representation is a convenient (and meaningful, as we will see later in detail) way of describing a general displacement. Given a displacement $g_T = ([R], \mathbf{d})$, we can create the 4×4 *homogeneous transform* as

$$[T] = \left[\begin{array}{ccc|c} & & & \\ & [R] & & \mathbf{d} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]. \tag{1.54}$$

This four-dimensional matrix contains all the information about the displacement. In order to match the dimensions, we need to add a fourth component to our vectors. Notice that we are transforming two types of things with a general displacement: we transform points (their position) and we also transform directions. Up until now, we have treated these two in the same manner, as three-dimensional vectors, even though we knew that they were different things. In the homogeneous representation, they will have also different expressions.

For transforming position of points, we will add a unit as the last element of the vectors, so that

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ 1 \end{Bmatrix} = [T] \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{Bmatrix}. \tag{1.55}$$

For transforming directions, we will add zero as the last element,

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ 0 \end{Bmatrix} = [T] \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{Bmatrix}. \tag{1.56}$$

We will cover the geometric explanation of this fourth component in subsequent sections dealing with projective geometry. For our purpose now, the addition of that last component allows us to operate a displacement by using a single matrix. The expressions found above for composition of displacements, inverse displacement and relative displacement hold, that is,

$$\begin{aligned}
&\text{Composition:} \quad \mathbf{X} = g_{T3}(\mathbf{x}) = g_{T1}(\mathbf{y})g_{T2}(\mathbf{x}) = [T_1][T_2]\mathbf{x}, \\
&\text{Inverse:} \quad \mathbf{x} = g_{T^{-1}}(\mathbf{X}) = (g_T)^{-1}(\mathbf{X}) = [T]^{-1}\mathbf{X}, \\
&\text{Relative displacement:} \quad \mathbf{X}_2 = g_{T12}(\mathbf{X}_1) = (g_{T2}g_{T1}^{-1})(\mathbf{X}_1) = [T_2][T_1]^{-1}\mathbf{X}_1.
\end{aligned} \tag{1.57}$$

Exercise: Check that the rotation matrix and displacement vector obtained with the 4×4 homogeneous matrix for the composition of displacements, inverse displacement and relative displacement are the same as the ones derived above.

We denote a rotation and a translation about and along an axis as a *screw displacement*. The screw displacements along the coordinate axes X, Y and Z take a simple form,

$$[X(\alpha, a)] = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad [Y(\beta, b)] = \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & b \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.58}$$

and

$$[Z(\theta, t)] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1.59}$$