1. In this problem you'll derive the minimum and maximum numbers of elements in a heap of height \( h \).

(a) What are the minimum number of nodes at level \( i \)? The root is at level 0, the children of root are at level 1, the grandchildren at level 2, etc. (Note: we’re not asking for the total number of nodes – just the number of nodes at a given level of the tree. Also, we assume a level with no nodes does not exist.)

(b) What are the maximum number of nodes at level \( i \)?

(c) Derive (showing your work) the minimum total number of nodes in a tree of height \( h \) (meaning the 0-based index of the last layer in the tree is \( h \)) using the summation symbol (\( \sum \)).

(d) Using properties in Appendix A of the textbook, give a closed-form solution (no \( \sum \) symbol) of the minimum total number of nodes.

(e) Derive the maximum total number of nodes in a tree of height \( h \) using the summation symbol (\( \sum \)).

(f) Give a closed-form solution of the maximum total number of nodes.

2. Prove that in any subtree of a max-heap, the root of the subtree contains the largest value occurring anywhere in that subtree. Assume a function \( \text{parent}^0(i) \) (recall functional iteration discussed in section 3.2 of the textbook). You will need to show that \( A[\text{parent}^0(i)] \geq A[i] \) forall \( j \geq 1 \). You'll do this using mathematical induction on \( j \).

3. Where in a max-heap might the smallest element reside, assuming that all elements are distinct?

4. Is an array that is in sorted order a min-heap?

5. Consider the array \( \langle 23, 17, 14, 6, 13, 10, 1, 5, 7, 12 \rangle \).

(a) Draw the tree associated with this array.

(b) Is this a max-heap? If not, circle the offending piece(s) in your drawing of the tree.

6. Using 1-based indices, show that, with the array representation for storing an \( n \)-element heap, the leaves are the nodes indexed by \( \lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \ldots, n \). Hint: one approach is to use the left-child and right-child functions, and show for what values of \( i \) the child indices are not valid, i.e. greater than \( n \).

7. Using figure 6.2 as a model (which uses 1-based indices), illustrate the operation of \( \text{max-heapify}(A, 3) \) on the array \( A = \langle 27, 17, 3, 16, 13, 10, 1, 5, 7, 12, 4, 8, 9, 0 \rangle \).

8. Using figure 6.3 as a model, illustrate the operation of \( \text{build-max-heap} \) on the array \( A = \langle 5, 3, 17, 10, 84, 19, 16, 22, 9 \rangle \).