1. In this problem you’ll derive the minimum and maximum numbers of elements in a heap of height $h$.

   (a) What are the minimum number of nodes at level $i$? The root is at level 0, the children of root are at level 1, the grandchildren at level 2, etc. \((\text{Note: we’re not asking for the total number of nodes – just the number of nodes at a given level of the tree.})\)

   (b) What are the maximum number of nodes at level $i$?

   (c) Derive (showing your work) the minimum total number of nodes in a tree of height $h$ using the summation symbol \(\sum\).

   (d) Using properties in Appendix A of the textbook, give a closed-form solution (no \(\sum\) symbol) of the minimum total number of nodes.

   (e) Derive the maximum total number of nodes in a tree of height $h$ using the summation symbol \(\sum\).

   (f) Give a closed-form solution of the maximum total number of nodes.

2. Prove that in any subtree of a max-heap, the root of the subtree contains the largest value occurring anywhere in that subtree. Assume a function parent\(^j\)\( i\) (recall functional iteration discussed in section 3.2 of the textbook). You will need to show that $A[\text{parent}\(^j\)\( i\)] \geq A[i]$. You’ll do this using mathematical induction on $j$.

3. Where in a max-heap might the smallest element reside, assuming that all elements are distinct?

4. Is an array that is in sorted order a min-heap?

5. Consider the array \(\langle 23, 17, 14, 6, 13, 10, 1, 5, 7, 12 \rangle\).

   (a) Draw the tree associated with this array.

   (b) Is this a max-heap? If not, circle the offending piece(s) in your drawing of the tree.

6. Using 1-based indices, show that, with the array representation for storing an $n$-element heap, the leaves are the nodes indexed by $\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \ldots, n$. \(\text{Hint: one approach is to use the left-child and right-child functions, and show for what values of } i \text{ the child indices are not valid, i.e. greater than } n.\)

7. Using figure 6.2 as a model, illustrate the operation of \texttt{max-heapify}(A,3) on the array $A = \langle 27, 17, 3, 16, 13, 10, 1, 5, 7, 12, 4, 8, 9, 0 \rangle$.

8. Using figure 6.3 as a model, illustrate the operation of \texttt{build-max-heap} on the array $A = \langle 5, 3, 17, 10, 84, 19, 16, 22, 9 \rangle$. 

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