Chapter 1: Signals and Linear Systems

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1.1 Preview

To study communication systems, 2 topics need to be understood:

A. Linear systems, in time and frequency domains
B. Probability and analysis of random signals

Most communication channels, sub-blocks of transmitters and receivers can be modeled as linear time-invariant (LTI) systems.

Therefore, linear system analysis methods can be used for analysis.

This Chapter covers Fourier series and transforms, power and energy, sampling theorem, and low-pass representation of band-pass signals.
1.2 Fourier Series

The input-output relation of a LTI system is given by the convolution operation, defined as:

\[ y(t) = x(t) * h(t) \]  \hspace{1cm} (1.2.1)

What is the convolution integral?

Here \( h(t) \) is the impulse response of the system, \( x(t) \) is the input signal, And \( y(t) \) is the output signal. If \( x(t) \) is a complex exponential given by

\[ x(t) = Ae^{j2\pi f_0 t} \]  \hspace{1cm} (1.2.2)

What will the output look like?
We see that the output signal \( y(t) \) is a complex exponential with the same frequency \( f_0 \) as the input signal \( x(t) \).

But the amplitude of \( y(t) \) is the amplitude of \( x(t) \) multiplied by _________.

Note: The above quantity is a function of the impulse response of the LTI system, \( h(t) \); and the frequency of the input signal \( x(t) \), \( f_0 \).

This greatly simplifies the computation of an LTI system’s response to exponential input signals.
Because of the above reason, it is natural to look for ways of expanding signals as sums of complex exponentials.

Fourier series and Fourier transforms are such techniques.

Fourier series (FS): Orthogonal expansion of periodic signals with period $T_0$ when the signal set $\{\sin(2\pi n t / T_0), \cos(2\pi n t / T_0)\}$ is used as the basis for the expansion.

With this basis, any periodic signal $x(t)$ with period $T_0$ can be expressed as follows:

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi nt / T_0}$$  \hspace{1cm} (1.2.4)

$X_n$’s are called the FS coefficients of $x(t)$, and are given by:

$$x_n = \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} x(t) e^{-j2\pi nt / T_0} \, dt$$  \hspace{1cm} (1.2.5)
In Eq. (1.2.5), $\alpha$ is an arbitrary constant chosen to simplify the integral computation. The frequency $f_0 = 1/T_0$ is the fundamental frequency of the periodic signal $x(t)$. The frequency $f_n = nf_0$ is the $n^{th}$ harmonic.

For most applications, $\alpha = 0$ or $-T_0/2$ is a good choice for $\alpha$.

This type of FS is known as the exponential FS (eFS). eFS is usable for both real-valued and complex-valued periodic signals. In general, even when $x(t)$ is a real-valued periodic signal, its FS coefficients $\{x_n\}$ are usually complex numbers. In this case, we have:

$$
\begin{align*}
\chi_{-n} &= \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} x(t) e^{j2\pi nt/T_0} \, dt \\
&= \frac{1}{T_0} \left[ \int_{\alpha}^{\alpha+T_0} x(t) e^{-j2\pi nt/T_0} \, dt \right]^* \\
&= x_n^*
\end{align*}
$$

(1.2.6)
From Eq. (1.2.6), we observe that

\[
\begin{align*}
|x_n| &= |x_{-n}| \\
\angle x_n &= -\angle x_{-n}
\end{align*}
\]  

(1.2.7)

This means that the FS coefficients of a real-valued signal possess Hermitian symmetry, i.e. their magnitude is even & their phase odd.

Another type of FS, known as *trigonometric Fourier series* (tFS), is only usable for real-valued periodic signals.

tFS can be obtained by first defining \(x_n\) and \(x_{-n}\) as follows:

\[
x_n = \frac{a_n - jb_n}{2} \\
x_{-n} = \frac{a_n + jb_n}{2}
\]  

(1.2.8)
With Eqs. (1.2.8) and (1.2.9), using Euler’s relation:

\[ e^{-j2\pi nt/T_0} = \cos \left( 2\pi t \frac{n}{T_0} \right) - j \sin \left( 2\pi t \frac{n}{T_0} \right) \]  

(1.2.10)

we obtain:

\[
a_n = \frac{2}{T_0} \int_{\alpha}^{\alpha+T_0} x(t) \cos \left( 2\pi t \frac{n}{T_0} \right) \, dt
\]

\[
b_n = \frac{2}{T_0} \int_{\alpha}^{\alpha+T_0} x(t) \sin \left( 2\pi t \frac{n}{T_0} \right) \, dt
\]  

(1.2.11)

and therefore arrive at:

\[
x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \left( 2\pi t \frac{n}{T_0} \right) + b_n \sin \left( 2\pi t \frac{n}{T_0} \right) \right)
\]  

(1.2.12)
From these Eqs., note that if \( n = 0 \), \( b_0 = 0 \) always, and thus \( a_0 = 2x_0 \).

By defining two more quantities

\[
\begin{align*}
c_n &= \sqrt{a_n^2 + b_n^2} \\
\theta_n &= -\arctan \frac{b_n}{a_n}
\end{align*}
\]

and using the property

\[
a \cos \phi + b \sin \phi = \sqrt{a^2 + b^2} \cos \left( \phi - \arctan \frac{b}{a} \right)
\]

We can re-write Eq. (1.2.12) as follows:

\[
x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n \cos \left( 2\pi t \frac{n}{T_0} + \theta_n \right)
\]
Eq. (1.2.15) is the 3rd form of the FS expansion for real-valued periodic Signals. Generally, the FS coefficients \( \{x_n\} \) for real-valued signals are related to \( a_n, b_n, c_n \) and \( \theta_n \) through the following equalities:

\[
\begin{align*}
a_n &= 2 \text{Re}[x_n] \\
b_n &= -2 \text{Im}[x_n] \\
c_n &= 2|x_n| \\
\theta_n &= \angle x_n
\end{align*}
\] (1.2.16)

Plots of \( |x_n| \) and \( \angle x_n \) vs. \( n \) or \( nf_0 \) are called the discrete spectra of \( x(t) \). Namely, they are called the magnitude spectrum and phase spectrum, respectively, for the input signal \( x(t) \).
If \( x(t) \) is real and even, i.e. if \( x(-t) = x(t) \), then taking \( \alpha = -\frac{T_0}{2} \), we get the following expression for \( b_n \):

\[
b_n = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \sin \left(2\pi t \frac{n}{T_0}\right) dt \tag{1.2.17}
\]

Eq. (1.2.17) is 0, because the sine function is an odd function of \( t \).

Therefore, for a real and even periodic signal \( x(t) \), all \( x_n \)'s are real. In this case the tFS consists of only cosine functions.

Similarly, if \( x(t) \) is real and odd, i.e. \( x(-t) = -x(t) \), then from Eq. (1.2.11)

\[
a_n = \frac{2}{T_0} \int_{\alpha}^{\alpha+T_0} x(t) \cos \left(2\pi t \frac{n}{T_0}\right) dt \tag{1.2.18}
\]

is 0; and all \( x_n \)'s are imaginary instead of real. In this case the tFS will consist of only sine functions.
1.2 Illustrative Problems

There are 3 IP’s, from pp. 4~12 of the textbook, covering Section1.2.
1.2.1 Periodic Signals & LTI Systems

When a periodic signal $x(t)$ is passed through an LTI system, as shown in Figure 1.9, the output signal $y(t)$ is also periodic and usually with the same period as $x(t)$. Exceptions? Thus $y(t)$ also has an FS expansion.

If $x(t)$ and $y(t)$ are defined as:

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi nt/T_0} \quad (1.2.26)$$

$$y(t) = \sum_{n=-\infty}^{\infty} y_n e^{j2\pi nt/T_0} \quad (1.2.27)$$
Then the relation between the FS coefficients of \( x(t) \) and \( y(t) \) can be obtained by using the convolution integral

\[
y(t) = \int_{-\infty}^{\infty} x(t - \tau) h(\tau) \, d\tau
\]

From Eq. (1.2.28) we have:

\[
y_n = x_n H \left( \frac{n}{T_0} \right)
\]
Where \( H(f) \) denotes the frequency response, a.k.a. transfer function, of the LTI system given as the Fourier transform of its impulse response \( h(t) \):

\[
H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi ft} \, dt \quad (1.2.30)
\]

1.2.1 Illustrative Problem

There is 1 IP, from pp. 13~16 of the textbook, covering Section 1.2.1.
1.3 Fourier Transforms

The Fourier transform (FT) is the extension of the FS to non-periodic signals.

The FT of a signal $x(t)$ is denoted by $X(f)$ or $\mathcal{F}[x(t)]$, and is defined by

$$\mathcal{F}[x(t)] = X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} \, dt$$  \hspace{1cm} (1.3.1)

The inverse FT of $X(f)$ is $x(t)$, given by

$$\mathcal{F}^{-1}[X(f)] = x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft} \, df$$  \hspace{1cm} (1.3.2)

If $x(t)$ is a real signal, then $X(f)$ satisfies the Hermitian symmetry, i.e.,

$$X(-f) = X^*(f)$$  \hspace{1cm} (1.3.3)
The FT has several important properties, summarized as follows.

1. Linearity: The FT of a linear combination of two or more signals is the combination of their corresponding FT’s.

\[
\mathcal{F}[\alpha x_1(t) + \beta x_2(t)] = \alpha \mathcal{F}[x_1(t)] + \beta \mathcal{F}[x_2(t)]
\]  
(1.3.4)

2. Duality: If \( X(f) = \mathcal{F}[x(t)] \), then

\[
\mathcal{F}[X(t)] = x(-f)
\]  
(1.3.5)

3. Time shift: A shift in the time domain results in a phase shift in the frequency domain. If \( X(f) = \mathcal{F}[x(t)] \), then

\[
\mathcal{F}[x(t - t_0)] = e^{-j2\pi f t_0} X(f)
\]  
(1.3.6)
4. Scaling: An expansion in the time domain results in a contraction in the frequency domain, and vice versa. If \( X(f) = \mathcal{F}[x(t)] \), then

\[
\mathcal{F}[x(at)] = \frac{1}{|a|} X \left( \frac{f}{a} \right), \quad a \neq 0 \tag{1.3.7}
\]

5. Modulation: Multiplication by an exponential in the time domain corresponds to a frequency shift in the frequency domain. If \( X(f) = \mathcal{F}[x(t)] \), then

\[
\begin{align*}
\mathcal{F}[e^{j2\pi f_0 t} x(t)] &= X(f - f_0) \\
\mathcal{F}[x(t) \cos(2\pi f_0 t)] &= \frac{1}{2} [X(f - f_0) + X(f + f_0)]
\end{align*} \tag{1.3.8}
\]

6. Differentiation: Differentiation in the time domain corresponds to multiplication by \( j2\pi f \) in the frequency domain. If \( X(f) = \mathcal{F}[x(t)] \), then

\[
\mathcal{F}[x'(t)] = j2\pi f X(f) \tag{1.3.9}
\]
6. Differentiation (continued): And similarly, for higher orders,

\[ \mathcal{F} \left[ \frac{d^n}{dt^n} x(t) \right] = (j2\pi f)^n X(f) \quad (1.3.10) \]

7. Convolution: Convolution in the time domain is equivalent to multiplication in the frequency domain, and vice versa.

If \( X(f) = \mathcal{F}[x(t)] \) and \( Y(f) = \mathcal{F}[y(t)] \), then

\[ \mathcal{F}[x(t) * y(t)] = X(f)Y(f) \quad (1.3.11) \]
\[ \mathcal{F}[x(t)y(t)] = X(f) * Y(f) \quad (1.3.12) \]

8. Parseval’s relation: If \( X(f) = \mathcal{F}[x(t)] \) and \( Y(f) = \mathcal{F}[y(t)] \), then

\[ \int_{-\infty}^{\infty} x(t)y^*(t) \, dt = \int_{-\infty}^{\infty} X(f)Y^*(f) \, df \quad (1.3.13) \]
\[ \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df \quad (1.3.14) \]

Eq. (1.3.14) is also known as Rayleigh’s relation.
Table 1.1 (on next slide) presents some useful FT pairs. In Table 1.1, $u_1(t)$ denotes the unit step function, $\delta(t)$ denotes the impulse signal. And $\text{sgn}(t)$ denotes the \textit{signum function} defined as:

$$\text{sgn}(t) = \begin{cases} 
1, & t > 0 \\
0, & t = 0 \\
-1, & t < 0 
\end{cases} \quad (1.3.15)$$

Also, $\delta^{(n)}(t)$ in the Table denotes the $n^{\text{th}}$ derivative of the impulse signal.
<table>
<thead>
<tr>
<th>$x(t)$</th>
<th>$X(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta(t)$</td>
<td>$1$</td>
</tr>
<tr>
<td>$1$</td>
<td>$\delta(f)$</td>
</tr>
<tr>
<td>$\delta(t - t_0)$</td>
<td>$e^{-j2\pi f t_0}$</td>
</tr>
<tr>
<td>$e^{j2\pi f_0 t}$</td>
<td>$\delta(f - f_0)$</td>
</tr>
<tr>
<td>$\cos(2\pi f_0 t)$</td>
<td>$\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$</td>
</tr>
<tr>
<td>$\sin(2\pi f_0 t)$</td>
<td>$\frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0)$</td>
</tr>
<tr>
<td>$\Pi(t)$</td>
<td>$\text{sinc}(f)$</td>
</tr>
<tr>
<td>$\text{sinc}(t)$</td>
<td>$\Pi(f)$</td>
</tr>
<tr>
<td>$\Lambda(t)$</td>
<td>$\text{sinc}^2(f)$</td>
</tr>
<tr>
<td>$\text{sinc}^2(t)$</td>
<td>$\Lambda(f)$</td>
</tr>
<tr>
<td>$e^{-\alpha t} u_{-1}(t), \quad \alpha &gt; 0$</td>
<td>$\frac{1}{\alpha + j2\pi f}$</td>
</tr>
<tr>
<td>$te^{-\alpha t} u_{-1}(t), \quad \alpha &gt; 0$</td>
<td>$\frac{1}{(\alpha + j2\pi f)^2}$</td>
</tr>
<tr>
<td>$e^{-\alpha</td>
<td>t</td>
</tr>
<tr>
<td>$e^{-\pi t^2}$</td>
<td>$e^{-\pi f^2}$</td>
</tr>
<tr>
<td>$\text{sgn}(t)$</td>
<td>$\frac{1}{j\pi f}$</td>
</tr>
<tr>
<td>$u_{-1}(t)$</td>
<td>$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$</td>
</tr>
<tr>
<td>$\delta'(t)$</td>
<td>$j2\pi f$</td>
</tr>
<tr>
<td>$\delta^{(n)}(t)$</td>
<td>$(j2\pi f)^n$</td>
</tr>
<tr>
<td>$\sum_{n=-\infty}^{\infty} \delta(t - nT_0)$</td>
<td>$\frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T_0})$</td>
</tr>
</tbody>
</table>

Table 1.1: Table of Fourier transform pairs
For a periodic signal \( x(t) \) with period \( T_0 \), whose FS coefficients are given by \( x_n \) -- that is, recalling Eq. (1.2.4):

\[
x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi nt/T_0}
\]  

(1.2.4)

The FT for \( x(t) \) is obtained, using Table 1.1, as follows:

\[
X(f) = \mathcal{F}[x(t)]
\]

\[
= \mathcal{F} \left[ \sum_{n=-\infty}^{\infty} x_n e^{j2\pi nt/T_0} \right]
\]

\[
= \sum_{n=-\infty}^{\infty} x_n \mathcal{F} \left[ e^{j2\pi nt/T_0} \right]
\]

\[
= \sum_{n=-\infty}^{\infty} x_n \delta \left( f - \frac{n}{T_0} \right)
\]  

(1.3.16)
Eq. (1.3.16) shows that the FT of a periodic signal $x(t)$ consists of impulses at multiples of the fundamental frequency (i.e. harmonics) of $x(t)$.

It is also possible to express the FS coefficients in terms of the FT of the truncated signal by

\[ x_n = \frac{1}{T_0} X_{T_0} \left( \frac{n}{T_0} \right) \]  \hspace{1cm} (1.3.17)

where $X_{T_0}(f)$ is the FT of $x_{T_0}(t)$, the truncated signal defined as

\[ x_{T_0}(t) = \begin{cases} x(t), & -\frac{T_0}{2} < t \leq \frac{T_0}{2} \\ 0, & \text{otherwise} \end{cases} \]  \hspace{1cm} (1.3.18)
The FT of a signal is called the *spectrum* of the signal. Typically, this is a complex function $X(f)$. Thus, to plot the spectrum usually requires both the magnitude spectrum $|X(f)|$ and the phase spectrum $\angle X(f)$.

1.3 Illustrative Problem

There is 1 IP, from pp. 20~22 of the textbook, covering Section 1.3.
1.3.1 Sampling Theorem

This theorem provides the critical link between continuous-time signals and discrete-time signals.

A band-limited signal, i.e. a signal whose FT vanishes for $|f| > W$ for some $W$, can be completely described in terms of its sample values taken at intervals $T_S$ as long as $T_S \leq 1/(2W)$. If the sampling is done at $T_S = 1/(2W)$, known as the Nyquist interval (or $1/(2T_S)$, the Nyquist rate), then the signal $x(t)$ can be reconstructed from the discrete sample values $\{ x[n] = x(nT_S) \}$, where $n = -\infty \sim \infty$, as shown below:

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_S) \text{sinc} \left( 2W(t - nT_S) \right) \quad (1.3.19)$$

based on the fact that the sampled waveform, $x_\delta(t)$, is defined as

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_S) \delta(t - nT_S) \quad (1.3.20)$$
Eq. (1.3.20) also has an FT, called $X_\delta(f)$, given by

$$X_\delta(f) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X\left(f - \frac{n}{T_s}\right) \quad \text{for all } f$$

$$= \frac{1}{T_s} X(f) \quad \text{for } |f| < W \quad (1.3.21)$$

Thus, passing $x_\delta(t)$ through an LPF with a bandwidth of $W$ and a gain of $T_s$ in the filter’s pass band will reproduce the original signal $x(t)$, since we can take the inverse FT on both sides of Eq. (1.3.21).

That is, $X_\delta(f)T_s = X(f)$ and therefore $x_\delta(t)T_s = x(t)$.

Hint: Eq. (1.3.19) is derived similarly to Eqs. (1.2.22). See IP 1.1
Figure 1.17 represents Eq. (1.3.19) for $T_S = 1$ and $\{x[n]\}$ for $n = -3 \sim 3$, given $\{x[n]\} = \{1,1,-1,2,-2,1,2\}$. Note that $T_S = 2W$ and $x[n] = x(nT_S)$. What does $x(t)$ whose wave form is shown in solid line, equal to?
The discrete Fourier transform (DFT) of the discrete-time sequence $x[n]$ is denoted by $X_d(f)$, given as

$$X_d(f) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi fnT_s} \quad (1.3.22)$$

Recall that Eq. (1.3.21) says $X(f) = T_sX_\delta(f)$. Thus, we conclude that

$$X(f) = T_sX_d(f) \quad \text{for } |f| < W \quad (1.3.23)$$

Eq. (1.3.23) relates the FT of an analog signal $x(t)$ to the DFT of its own corresponding sampled signal, for $|f| < W$.

DFT is computed using the famous fast Fourier transform (FFT) method.
The FFT algorithm

A sequence of $N$ samples of $x(t)$, taken at intervals of $T_S$, is used.

This results in a sequence of $N$ samples of $X_d(f)$ in the frequency interval $[0, f_S]$ where $f_S = 2W = 1/T_S$, is the Nyquist rate.

When the $N$ samples are $\Delta f = f_S/N$ apart, value of $\Delta f$ gives the frequency resolution of the resulting FT.

The FFT algorithm is computationally efficient if $N$ is a power of 2. In many cases when this is not the case, $N$ will be made a power of 2 by techniques such as zero padding.

The FFT algorithm produces the DFT of the sampled signal, i.e. $X_d(f)$. This means Eq. (1.3.23) must be used to get $X(f)$. 
The FFT MATLAB implementation

The MATLAB function fftseq.m takes as its input a time sequence $m$, the sampling interval $t_S$, and the required frequency resolution $df$. It returns a sequence whose length is a power of 2, the FFT of this sequence $M$, and the resulting frequency resolution. Note that the fftseq.m function code contains an fft(m,n) call.

See IP 1.6 for an application of this function.
1.3.2 Frequency-Domain Analysis of LTI Systems

The output $y(t)$ of an LTI system with impulse response $h(t)$ and a given input $x(t)$ is specified by the convolution integral:

$$y(t) = x(t) * h(t)$$  \hspace{1cm} (1.3.28)

Applying the convolution theorem, we obtain

$$Y(f) = X(f)H(f)$$  \hspace{1cm} (1.3.29)

where

$$H(f) = \mathcal{F}[h(t)] = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft} \, dt$$  \hspace{1cm} (1.3.30)

is the transfer function of the LTI system.
Eq. (1.3.29) can be alternatively written in terms of the magnitude and phase spectra as follows:

$$\begin{align*}
|Y(f)| &= |X(f)| \cdot |H(f)| \\
\angle Y(f) &= \angle X(f) + \angle H(f)
\end{align*}$$

(1.3.31)

See IP 1.7 (pp. 28–31) on LTI system analysis in the frequency domain.
1.4 Power and Energy

The *energy* and *power* of a real signal \( x(t) \), denoted as \( E_X \) and \( P_X \), are defined as

\[
\begin{align*}
E_X &= \int_{-\infty}^{\infty} x^2(t) \, dt \\
P_X &= \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) \, dt
\end{align*}
\]

A signal with finite energy is called an *energy signal*. A signal with positive and finite power is called a *power signal*.

\( x(t) = \Pi(t) \) is an energy signal. \( x(t) = \cos(t) \) is a power signal. In fact, all periodic signals are power signals.

The *energy spectral density* (ESD) of an energy signal \( \equiv \) the distribution of energy at various frequencies of the signal.
The ESD of a signal is given by

\[ \mathcal{G}_X(f) = |X(f)|^2 \] \hspace{1cm} (1.4.2)

And hence,

\[ E_X = \int_{-\infty}^{\infty} \mathcal{G}_X(f) \, df \] \hspace{1cm} (1.4.3)

Using the convolution theorem, we obtain

\[ \mathcal{G}_X(f) = \mathcal{F}[R_X(\tau)] \] \hspace{1cm} (1.4.4)

where \( R_X(\tau) \) is the *autocorrelation function* of real signal \( x(t) \), defined as

\[ R_X(\tau) = \int_{-\infty}^{\infty} x(t)x(t + \tau) \, dt \]

\[ = x(\tau) \ast x(-\tau) \] \hspace{1cm} (1.4.5)
For power signals, we define the *time-average autocorrelation function*

\[ R_X(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t + \tau) \, dt \]  

(1.4.6)

and the *power spectral density* (PSD)

\[ S_X(f) = \mathcal{F}[R_X(\tau)] \]  

(1.4.7)

Hence the total power is the integral of the PSD, given by

\[ P_X = \int_{-\infty}^{\infty} S_X(f) \, df \]  

(1.4.8)
A periodic signal \( x(t) \) with period \( T_0 \) and FS coefficients \( x_n \) has a PSD

\[
S_X(f) = \sum_{n=-\infty}^{\infty} |x_n|^2 \delta \left( f - \frac{n}{T_0} \right)
\]  

(1.4.9)

This means that all the power of \( x(t) \) is concentrated at the harmonics of the fundamental frequency; and the power at the \( n^{th} \) harmonic \( (n/T_0) \) is \( |x_n|^2 \), i.e. the magnitude square of its FS coefficient.

When \( x(t) \) passes through a filter with transfer function \( H(f) \), the output ESD and PSD are obtained as

\[
\begin{align*}
G_Y(f) &= |H(f)|^2 G_X(f) \\
S_Y(f) &= |H(f)|^2 S_X(f)
\end{align*}
\]  

(1.4.10)
If the discrete-time (sampled from $x(t)$) signal is used, the energy and power relation equivalents to Eq. (1.4.1) are

\[
\begin{align*}
E_X &= T_s \sum_{n=-\infty}^{\infty} x^2[n] \\
P_X &= \lim_{N \to \infty} \frac{1}{2N + 1} \sum_{n=-N}^{N} x^2[n] 
\end{align*}
\]  
(1.4.11)

Compare Eq. (1.4.11) to Eq. (1.4.1)

\[
\begin{align*}
E_X &= \int_{-\infty}^{\infty} x^2(t) \, dt \\
P_X &= \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) \, dt 
\end{align*}
\]  
(1.4.1)
Furthermore, if FFT is used, i.e., the length of the sequence is finite and the sequence is repeated, then

\[ E_X = T_s \sum_{n=0}^{N-1} x^2[n] \]
\[ P_X = \frac{1}{N} \sum_{n=0}^{N-1} x^2[n] \]

(1.4.12)

If \( X_d(f) \) is the DFT of the sequence \( x[n] \), then the ESD of \( x(t) \) can be obtained by using Eq. (1.3.23), and we arrive at

\[ g_X(f) = T_s^2 |X_d(f)|^2 \]

(1.4.13)

where \( T_s \) is the sampling interval. Nice and clean!
1.5 Lowpass Equivalent of Bandpass Signals

Bandpass signal: A signal all of whose frequency components are located in the neighborhood of a central frequency $f_0$ (and $-f_0$).

This means that for such a signal, $X(f) \equiv 0$ for $|f \pm f_0| > W$, and $W << f_0$.

Using this definition, we note that a lowpass signal is a signal whose frequency components are located around $0$, i.e. for $|f| > W$, $X(f) \equiv 0$.

Given a bandpass signal $x(t)$, we can define the analytic signal $z(t)$, whose FT is given as

$$ Z(f) = 2u_{-1}(f)X(f) $$

(1.5.1)

where $u_{-1}(f)$ is the unit step function.
Eq. (1.5.1) can be expressed in the time domain as

\[ z(t) = x(t) + j\hat{x}(t) \]  \hspace{1cm} (1.5.2)

where \( x'(t) \) is the Hilbert transform of \( x(t) \), defined as \( x'(t) = x(t) * (1/\pi t) \) in the time domain. In the frequency domain, it is given by

\[ \hat{X}(f) = -j \text{sgn}(f)X(f) \]  \hspace{1cm} (1.5.3)

The Hilbert transform function in MATLAB, hilbert.m, generates the complex sequence \( z(t) \). \( \text{Re}[z(t)] \) is the original sequence, while \( \text{Im}[z(t)] \) is the Hilbert transform of the original sequence. The lowpass equivalent of \( x(t) \), denoted by \( x_l(t) \), is expressed in terms of \( z(t) \) as

\[ x_l(t) = z(t)e^{-j2\pi f_0 t} \]  \hspace{1cm} (1.5.4)
From Eq. (1.5.4) we can obtain

\[
\begin{align*}
\dot{x}(t) &= \Re [\dot{x}_l(t) e^{j2\pi f_0 t}] \\
\hat{x}(t) &= \Im [\dot{x}_l(t) e^{j2\pi f_0 t}]
\end{align*}
\] (1.5.5)

In the frequency domain we have

\[
X_l(f) = Z(f + f_0) = 2u_{-1}(f + f_0)X(f + f_0)
\] (1.5.6)

\[
X_l(f) = X(f - f_0) + X^*(-f - f_0)
\] (1.5.7)

The lowpass equivalent of a real bandpass signal is generally a complex signal. Its real part, denoted by \(x_c(t)\), is called the *in-phase component* of \(x(t)\). Its imaginary part, \(x_s(t)\), is called the quadrature component of \(x(t)\). That is,

\[
x_l(t) = x_c(t) + j x_s(t)
\] (1.5.8)
From Eqs. (1.5.4) through (1.5.8) we arrive at

\[
\begin{align*}
    x(t) &= x_c(t) \cos(2\pi f_0 t) - x_s(t) \sin(2\pi f_0 t) \\
    \hat{x}(t) &= x_s(t) \cos(2\pi f_0 t) + x_c(t) \sin(2\pi f_0 t)
\end{align*}
\]  \hspace{1cm} (1.5.9)

Expressing \(x_l(t)\) in polar coordinates,

\[
x_l(t) = V(t)e^{j\Theta(t)}
\]  \hspace{1cm} (1.5.10)

where \(V(t)\) is the envelope and \(\Theta(t)\) is the phase of \(x(t)\). Thus, we can express \(x(t)\) in terms of \(V(t)\) and \(\Theta(t)\) as

\[
x(t) = V(t) \cos(2\pi f_0 t + \Theta(t))
\]  \hspace{1cm} (1.5.11)
From the concepts we learned before, $V(t)$ and $\Theta(t)$ can be expressed as

$$\begin{align*}
V(t) &= \sqrt{\chi_c^2(t) + \chi_s^2(t)} \\
\Theta(t) &= \arctan \frac{\chi_s(t)}{\chi_c(t)}
\end{align*}$$

Eq. (1.5.12)

or equivalently,

$$\begin{align*}
V(t) &= \sqrt{x^2(t) + \hat{x}^2(t)} \\
\Theta(t) &= \arctan \frac{\hat{x}(t)}{x(t)} - 2\pi f_0 t
\end{align*}$$

Eq. (1.5.13)

Eq. (1.5.13) shows that $V(t)$ does not depend on the central frequency $f_0$, but $\Theta(t)$ does. See the 4 functions `analytic.m`, `loweq.m`, `quadcomp.m` and `env_phas.m` for generating these signals and components. Then go over IP 1.9 starting on p. 37 of the textbook.