7.3 Stress, Deformation, Conservation, and Rheology: A Survey of Key Concepts in Continuum Mechanics

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7.3.1 Introduction

Conservation laws

The concept in physics that any system evolves in such a way as to maintain equilibrium in the balance of a measurable physical quantity. In continuum mechanics, conservation equations typically state that the rate of change of some quantity, such as mass, momentum, or energy, within a definable volume is balanced by the amount of the quantity produced (or consumed) within the volume and the rate at which the quantity is transported across the volume boundaries.

Constitutive equations

Mathematical expressions of the behavior of idealized materials that describe the relation between physical quantities, such as stresses and strains or stresses and strain rates.

Continuum mechanics

The branch of mechanics that deals with the analysis of an idealized matter that completely fills the space it occupies and has properties that, when averaged over appropriate spatial scales, vary continuously across that space.

Newton

A unit of measure of force, defined as kilogram meters per squared second (kg m s⁻²).

Pascal

A unit of measure of stress, defined as newtons per square meter (N m⁻²).
### Abstract

This chapter provides a brief survey of key concepts in continuum mechanics. It focuses on the fundamental physical concepts that underlie derivations of the mathematical formulations of stress, strain, hydraulic head, pore-fluid pressure, and conservation equations. It then shows how stresses are linked to strain and rates of distortion through some special cases of idealized material behaviors. The goal is to equip the reader with a physical understanding of key mathematical formulations that anchor continuum mechanics in order to better understand theoretical studies published in geomorphology.

### 7.3.1 Introduction

Quantitative understanding of the physical processes that drive landscape evolution – from tectonic uplift to erosional processes that gnaw the land – is firmly rooted in the concepts of stress, deformation, rates of deformation, conservation, and rheology, which lie at the heart of continuum mechanics. The idea that a basic understanding of continuum mechanics is important for understanding physical processes in geoscience is shared by many and has been advocated for at least several decades (e.g., Johnson, 1970). Explicit applications of physics to geomorphic problems date back to at least the late 1800s, when GK Gilbert used basic concepts of force balance to explain intrusions in the Henry Mountains, Utah (Gilbert, 1877). However, routine treatment of these topics in dedicated courses taught in geoscience or environmental science departments at the undergraduate, and even the graduate, level remains elusive. A few textbooks have focused on fundamental concepts of continuum mechanics with applications to geological problems (e.g., Johnson, 1970; Means, 1976; Middleton and Wilcock, 1994; Turcotte and Schubert, 2002; Cuffey and Paterson, 2010), and recently a textbook has emerged that is focused on the mechanics of landscapes (Anderson and Anderson, 2010).

In this chapter, a summary of basic concepts of continuum mechanics is presented, hopefully with enough transparency that the reader can grasp the fundamental, underlying physics. The purpose of writing this chapter is to distill the concepts presented more broadly in textbooks into a concise package. Consequently, our treatment of these concepts is necessarily brief. A comprehensive treatment of continuum mechanics, but a challenging read, is provided by Malvern (1969). Thorough and approachable discussions targeted at students of Earth science are provided by Means (1976), Middleton and Wilcock (1994), Turcotte and Schubert (2002), and Anderson and Anderson (2010). Bird et al. (1960) provided a lucid discussion of fundamental concepts related to fluid mechanics, and Bejan (1984) presented an excellent treatment of the application of scaling analysis to problems in fluid mechanics.

Discussion here is focused on key concepts in continuum mechanics that a reader needs to grasp in order to understand and evaluate theoretical papers in the geomorphic, and more broadly the geologic, literature. After presenting concepts of force, stress, deformation, strain, and rate of deformation, the chapter discusses conservation laws of mass and momentum and then discusses the properties of a few special cases of idealized materials and how they respond to applied stresses. Constitutive equations, expressions that relate stresses to strains and rates of distortion, provide the linkage to the rheological behavior of materials (or how they deform and flow) under applied loads. This may be the first exposure to this material for some readers, so to make the concepts discussed here more concrete, the chapter concludes with an example application from the literature that relies on many of the concepts presented.

In a survey of this type, certain topics are treated superficially or omitted altogether. Unlike more comprehensive treatments in textbooks, discussions of material strength and methods to measure strength, stress, and strain are omitted, as are discussions of stress and strain ellipsoids, the concept of Mohr’s circle, and complex idealized materials. The interested reader can find excellent discussions of these topics in Selby (1982), Middleton and Wilcock (1994), and Turcotte and Schubert (2002). Detailed Earth science applications of the concepts discussed here occur, for example, in Johnson (1970), Middleton and Wilcock (1994), Turcotte and Schubert (2002) and Cuffey and Paterson (2010), and applications focused on geomorphology are presented in Anderson and Anderson (2010).

The discussion presented here is mathematical by necessity. However, the basic mathematical concepts discussed are not extraordinarily difficult. Much of the mathematical development of stress and strain is geometrical in nature and relatively easy to conceptualize in one or two dimensions. A few slightly more sophisticated mathematical concepts are

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**Rheology** The science of the deformation and flow of matter.

**Scalar** A physical quantity defined solely by its magnitude.

**Strain** The distortion of a body, such as a change in length or a change in angle between perpendicular elements, caused by application of a stress.

**Stress** Surface force per unit area exerted on a plane having a particular orientation within or on a body of material.

**Tensor** A physical quantity defined by its magnitude, direction of operation, and position in space.

**Vector** A physical quantity defined by its magnitude and direction of operation.
used in some derivations, such as limits and gradients, but they, too, are easily grasped. Although most derivations of concepts are presented in one or two dimensions, we live in a three-dimensional world and stresses, strains, and fluxes are three-dimensional in nature. In some places, it is, therefore, convenient to extend the mathematical presentation from one or two dimensions to three dimensions through the use of vector and tensor notation. The chapter generally avoids the compact, but cryptic, notation associated with vector and tensor operations used by many authors of research papers, except in a few explicit instances. When a more compact notation is convenient to the presentation, it is linked explicitly to an example of a fully developed component of an equation. By following this strategy, we hope to demystify some of the mathematical description of the concepts discussed, with the goals of providing the reader a direct linkage between the conceptual mathematics and the underlying physical principles, and an entrance to the theoretical literature published with regard to geomorphic processes. Thorough treatments of vector and tensor notation and mathematical concepts such as divergence, gradient, curl, dot products, and dyadic products can be found, for example, in Malvern (1969), Schey (1973), Means (1976), and Middleton and Wilcock (1994). In this chapter, scalar quantities (those defined solely by their magnitude) are represented by a single letter (e.g., \( m \)), vectors (quantities having magnitude and direction) are represented by an underscore (e.g., \( \mathbf{L} \)), and tensors (quantities defined by magnitude, direction, and position) by an overlying tilde (e.g., \( \mathbf{\tilde{s}} \)) or with indicial notation (e.g., \( \sigma_{ij} \)).

### 7.3.2 Continuum

When applying concepts of mechanics to problems in Earth sciences, we are interested in forces applied to, and deformation and flow of, idealized continuous bodies (a continuum). But what constitutes a continuum; is such an idealization appropriate or adequate? A continuum is an idealized region of space filled with matter having properties that, when averaged over appropriate spatial scales, vary continuously across that space. Under this simplifying concept, we disregard molecular structure of material. At the macroscopic scale, we ignore discontinuities (e.g., discontinuities between mineral grains or fractures in rock bodies) and assume that material can be adequately characterized by averaged properties. Clearly, such idealized matter does not exist, as discontinuities are present at virtually all scales. In some instances, discontinuities may place bounds on the region that can be considered a continuum. However, if a scale appropriate for the problem is selected, discontinuities at a smaller scale can be tolerated, and average properties of the matter can be assumed to vary smoothly across the scale of interest.

### 7.3.3 Force

Newton’s second law states that the time rate of change of momentum of a body is proportional to the sum of the forces acting upon that body (e.g., Johnson, 1970; Middleton and Wilcock, 1994). Because momentum is defined as the product of mass and velocity, Newton’s second law can be written as

\[
F = \frac{d(mv)}{dt}
\]

where \( F \) represents the forces acting on the body, \( \frac{d}{dt} \) the total derivative that represents rate of change with time, \( m \) the mass of the body, and \( v \) its velocity. For constant mass, this expression becomes

\[
F = m \frac{dv}{dt}
\]

and because \( \frac{dv}{dt} \) is the definition of acceleration, we get the familiar expression

\[
F = ma
\]

From the above expression, we see that force has a unit of kg m s\(^{-2}\), which is called a newton (N).

In mechanics, two classes of forces can be defined: body forces and surface forces. Body forces act equally on every element of mass within a body and are proportional to its mass or volume (Middleton and Wilcock, 1994; Turcotte and Schubert, 2002). An example of a body force is the force of gravity; the weight of an element is the product of its mass and the acceleration of gravity, \( g \). Surface, or contact, forces, on the other hand, act on the bounding surface of a body. Unlike a body force, the influence of a surface force is proportional to the size of the area over which it acts; furthermore, it acts in a specific direction and in a specific position (e.g., Middleton and Wilcock, 1994; Turcotte and Schubert, 2002). An example of surface forces acting on a body can be illustrated by envisioning a person pushing a box across a table. As the box is pushed, there is not only a force acting perpendicular to the face being pushed, but also a component of that force acting tangentially along the surface of the box in contact with the table. As long as the magnitude of the tangential force exceeds the force of friction resisting sliding the box will slide. We can also envision that a box having a small footprint might be easier to push than one having a large footprint, because the magnitude of the tangential force transmitted to the base of the small box is focused across a smaller area and can more easily overcome the frictional force resisting sliding. Because the influence of a surface force is proportional to the size of the area over which it acts, an inherent geometric effect influences the changes in a body caused by that force. Therefore, having a common way of quantifying the effect of a surface force regardless of the size of the area over which it acts is useful. The best way to remove the geometric effect of a surface force acting on a box is to normalize the force by the area over which it acts, which leads to the concept of stress.

### 7.3.4 Stress

Stress, by definition, is the surface force per unit area exerted on a body of material and is given in units of newtons per square meter (N m\(^{-2}\)), or pascals (Pa). Stress is a very useful concept for understanding the impact a surface force has on a body. Intuitively, the more broadly a surface force is...
distributed, the lower the stress. For example, this is the underlying physics behind snow shoes. A person trying to walk across a soft, snow-covered surface commonly punches holes through, and sinks below, the surface. With snow shoes, however, that person’s weight (the imposed surface force) is distributed over a broader area of snow surface, and he or she is able to stay atop the surface because of the lower stress imparted. The examples thus far illustrate surface forces acting over a finite area. The broader concept of stress with respect to continuum mechanics, however, is related to the stress acting at a point or the force acting on an area in the limit as the size of the area diminishes to zero (e.g., Malvern, 1969; Middleton and Wilcock, 1994).

### 7.3.4.1 Total Stress

Stress is conveniently resolved into two components: a normal stress, which acts perpendicular to a surface, and a shear stress, which acts tangential to a surface. Figure 1 shows the normal and shear stresses acting on an elemental control volume defined in accordance with the Cartesian x, y, z coordinate system. Each stress is identified using a pair of subscripts: the first subscript refers to the direction perpendicular to the surface on which the stress is acting and the second subscript refers to the direction of the stress. Thus, normal stresses have identically paired subscripts, whereas shear stresses have unequal subscripts. Three stresses are defined on each surface perpendicular to a coordinate direction: a normal stress acting perpendicular to the surface and two orthogonal shear stresses acting along the surface. Hence, nine different stresses act on the three-dimensional volume, and assuming there is no acceleration, stresses equal in magnitude but acting in opposing directions are imposed on the other three faces. The forces associated with the normal stresses act to stretch or compress the elemental volume, whereas the forces associated with the shear stresses attempt to distort and rotate the elemental volume about each axis. For an element in equilibrium (not accelerating), the moments about each axis must balance. Therefore, the shear stresses are symmetric; for example, \( \sigma_{xy} = \sigma_{yx} \). Whereas nine stresses are defined in three dimensions, only six of those stresses are independent.

To properly account for the direction in which the stresses act, a sign convention for positive and negative stresses must be defined. Here, a common convention that outward-directed normal stresses are positive and inward-directed normal stresses are negative is adopted. Thus, normal stresses are defined to be positive in tension and negative in compression. Although common, this sign convention is not universal. For example, in soil mechanics it is common to consider compressive stresses positive, because compression is the most common state of soils dealt with by geotechnical engineers (e.g., Lambe and Whitman, 1969). Despite the seeming convenience of aligning the positive sign convention with the most common state of stress in Earth, there are both mathematical and physical reasons that trump this convenience. Mathematically, the outward-normal direction on the Cartesian elemental volume has a positive sense of direction on a positive face and a negative sense of direction on a negative face (see Figure 1). The normal stresses acting in those positive senses, thus, tend to pull the element in opposite directions leading to a state of tensile stress. The adopted sign convention also is consistent with physical changes that occur during normal strain, in which elongation associated with tension (a positive value because the ending state is longer than the starting state) is defined as positive strain. In classical thermodynamics, expansion of a gas is defined as doing positive work on its surrounding environment. Shear stresses are defined as positive if they act in a positive direction on a positive face, and in a negative direction on a negative face (Figure 1).

Stresses acting at a point can be represented in mathematical form as \( \sigma_{ij} \). This mathematical notation, in which \( i \) and \( j \) represent the coordinate axes as numbers (1, 2, 3) or letters (x, y, z), defines a stress matrix, also known as a stress tensor:

\[
\sigma_{ij} = \begin{pmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{pmatrix}
\]

\[4\]

In this matrix, the terms along the diagonal represent the normal stresses and the off-diagonal terms represent the shear stresses. Because pairs of shear stresses must be equal for an element in equilibrium, \( \sigma_{ij} = \sigma_{ji} \).

We can use the stress matrix to define the mean normal stress acting on the element as

\[
\bar{\sigma} = \frac{1}{3} (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})
\]

\[5\]

The mean normal stress can, furthermore, be equated with a mechanical mean pressure acting on the element. Because pressure is typically defined as positive in compression, we can define the mechanical mean pressure as

\[
\bar{p} = \frac{1}{3} (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})
\]

\[6\]
This expression tells us that a positive mean normal stress (tension) is equivalent to a negative mechanical mean pressure, whereas a negative mean normal stress (compression) is equivalent to a positive mechanical mean pressure.

In some analyses, it is useful to separate the mean normal stress (pressure) acting on a medium from the overall stress by subtracting the mean normal stress from the total stress. Separating stresses in this manner allows us to isolate explicitly those stresses that deviate from the mean stress. For an incompressible material, the stresses that deviate from the mean normal stress are those that cause deformation. Hence, the 'deviatoric stress' is defined as the difference between the total stress and the mean normal stress (e.g., Engelder, 1994). The deviatoric stress matrix can be written as

\[
\sigma_{ij}^D = \begin{pmatrix} 
\sigma_{xx} - (\bar{\sigma}) & \sigma_{xy} & \sigma_{xz} \\
\sigma_{xy} & \sigma_{yy} - (\bar{\sigma}) & \sigma_{yz} \\
\sigma_{xz} & \sigma_{yz} & \sigma_{zz} - (\bar{\sigma}) 
\end{pmatrix}
\]  

From this matrix, we see that only the normal stresses are affected; shear stresses are unaffected by variations in mean normal stress (Middleton and Wilcock, 1994; Turcotte and Schubert, 2002). Clearly, another way to write the total stress tensor is \( \sigma_{ij} = -\delta_{ij} \sigma_0 + \sigma_{ij}^D \), where \( \delta_{ij} \) is the identity matrix, known as the Kronecker delta, which is equal to 1 when \( i = j \) and zero otherwise.

### 7.3.4.2 Stresses Acting on Arbitrarily Oriented Surfaces

Although defining stresses coincident with defined coordinate directions is convenient, an ability to define stresses along surfaces oriented arbitrarily with respect to an original coordinate system is necessary to make full use of the concept of stress. Resolving stresses in a new coordinate system that is rotated with respect to an original coordinate system involves trigonometric manipulations of force balances, which involve directional sines and cosines of the angle of rotation. Detailed discussions of these manipulations are provided in Malvern (1969), Means (1976), Middleton and Wilcock (1994), and Turcotte and Schubert (2002). For stresses defined as positive in tension and acting within the xy-plane associated with a Cartesian coordinate system, the results are

\[
\sigma_{xx} = \sigma_{xx}' \cos^2 \theta + \sigma_{yy}' \sin^2 \theta + \sigma_{xy}' \sin 2\theta \]  

\[
\sigma_{yy} = \sigma_{xx}' \sin^2 \theta + \sigma_{yy}' \cos^2 \theta - \sigma_{xy}' \sin 2\theta \]  

\[
\sigma_{xy} = \frac{1}{2} (\sigma_{xx} - \sigma_{yy}) \sin 2\theta - \sigma_{xy}' \cos 2\theta \]  

In these expressions, the prime notation refers to stresses acting on surfaces that are oriented arbitrarily, and the unprimed notation refers to stresses acting on surfaces oriented with respect to the originally defined coordinate system. For \( \theta = 0 \), \( \sigma_{xx}' = \sigma_{xx}, \sigma_{yy}' = \sigma_{yy}, \) and \( \sigma_{xy}' = \sigma_{xy}, \) which is required if the coordinate systems are coincident. If we now suppose that a condition exists such that there are surfaces on which the shear stresses disappear, we can find the orientations of those surfaces by setting \( \sigma_{xy}' \) equal to zero in eqn [10] and solving for \( \theta \). That condition yields

\[
\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}} \]  

The direction \( \theta \) defined by eqn [11] is known as an axis of principal stress. Because \( \theta \) satisfies eqn [11], \( \theta + 90^\circ (\theta + \pi/2) \) must also satisfy eqn [11] (because \( \tan \pi = 0 \)). Hence, two orthogonal surfaces exist (in two dimensions) for which the shear stress is equal to zero. The normal stresses acting on those two planes are known as the principal stresses; for one plane, the normal stress achieves a maximum value known as the maximum principal stress (commonly signified as \( \sigma_1 \)), and for the other it achieves a minimum value known as the minimum principal stress (signified as \( \sigma_2 \) when considering stresses in two dimensions, or \( \sigma_3 \) when considering stresses in three dimensions). For the sign convention adopted, the maximum principal stress is the 'most positive' stress. Although this terminology is somewhat contradictory with most Earth science convention (in which the greatest compressive stress, here the 'most negative' normal stress, is considered the maximum principal stress), this is merely a matter of sign convention and has no bearing on the underlying physical concept.

Convenient expressions to have are those that relate normal and shear stresses for any arbitrary coordinate system in terms of the principal stresses and the angle of the new coordinate system with respect to the principal stress axes. For stresses in two dimensions having an \( xy \) coordinate system oriented at an angle \( \theta \) to the principal stress axes, the stresses \( \sigma_{xx}, \sigma_{yy} \) and \( \sigma_{xy} \) are related to the principal stresses and principal stress axes by (e.g., Middleton and Wilcock, 1994; Turcotte and Schubert, 2002)

\[
\sigma_{xx} = \frac{\sigma_1 + \sigma_2}{2} + \frac{(\sigma_1 - \sigma_2)}{2} \cos 2\theta \]  

\[
\sigma_{yy} = \frac{\sigma_1 + \sigma_2}{2} - \frac{(\sigma_1 - \sigma_2)}{2} \cos 2\theta \]  

\[
\sigma_{xy} = \frac{1}{2} (\sigma_1 - \sigma_2) \sin 2\theta \]  

where \( \theta \) is the angle between the direction of \( \sigma_1 \) and the direction of \( \sigma_{xx} \).

To find the surfaces on which shear stress \( \sigma_{xy} \) is a maximum value, differentiate eqn [10] with respect to \( \theta \) and set the result equal to zero. Doing so gives

\[
\tan 2\theta = \frac{\sigma_{yy} - \sigma_{xx}}{2\sigma_{xy}} \]  

Comparing eqn [15] with eqn [11] shows that the angle \( 2\theta \) for the principal axis orientation and \( 2\theta \) for the maximum shear stress orientation are negative reciprocals. This means that the axes that maximize shear stress lie at 45° to the axes of
principal stresses (Turcotte and Schubert, 2002). Setting \( \theta = \pi/4 \) in eqn [14] yields

\[
(s_{ij})_{max} = \sigma_1 - \sigma_2 \over 2
\]

Tensor notation expressions for three-dimensional stresses acting on planes of arbitrary direction relative to some defined coordinate direction are discussed in detail in Means (1976) and Middleton and Wilcock (1994).

Thus far, discussion of stresses has been implicitly restricted to homogenous, nonporous media. In porous media, stresses causing deformation can be influenced by the pressure of the fluid that fills the pores. That influence necessitates discussion of pore fluid, pore-fluid pressure, and the concept of effective stress. Because the most common fluid in porous material at Earth’s surface is water, the discussion below is restricted to pore water and pore-water pressure.

### 7.3.4.3 Pore Water, Hydraulic Head, and Pore-Water Pressure

Regolith (the mantle of fragmented material that overlies bedrock) and highly fractured rock at Earth’s surface—here termed soil—contain voids (pores) that are variously wetted or filled with water (pore water). Forces acting on pore water establish gradients of fluid potential, the work required to move a unit quantity of fluid from a datum to a specified position, and pore water flows in response to these gradients.

The concept of hydraulic head, a measure of the energy in a fluid-filled porous medium, usefully describes pore-water potential. Total hydraulic head, or potential per unit weight of fluid, can be defined in terms of two fundamental forms of energy: potential energy, defined in terms of gravitational and pressure potential energy, and kinetic energy, the energy associated with fluid motion (Hubbert, 1940; Ingebritsen and Sanford, 1998). In a typical soil subject to Darcian (seepage) flow, the flow velocity is usually very low and the kinetic energy is negligibly small compared to the total potential energy. Thus, for an incompressible fluid (fluid having a constant density; \( \rho_w \) for water) the total hydraulic head \( (h) \) in a water-filled soil is given by (Hubbert, 1940; Ingebritsen and Sanford, 1998)

\[
h = \psi + \frac{p}{\rho_w g}
\]

where \( \psi \) is the gravitational, or elevation, potential, and \( p/\rho_w g \) the pressure potential, in which \( p \) is the gauge pressure of the pore water relative to atmospheric pressure and \( g \) gravitational acceleration in the coordinate direction. Pore-water pressure, therefore, constitutes one of the two dominant components of the fluid potential in soils.

Pore-water pressure is isotropic, meaning that it is the same magnitude in all directions, but it varies with position relative to the water table within a soil (the depth horizon where pore-water pressure is atmospheric, which defines the zero-pressure datum) and with the proportion of soil weight carried by contacts among the soil grains (intergranular contacts). Below the water table, pore-water pressure is greater than atmospheric and positive; above the water table, pore-water pressure is less than atmospheric and negative owing to tensional capillary forces exerted on pore water (e.g., Remson and Randolph, 1962). If soil is saturated and water statically fills pore space, then the pore-water pressure is hydrostatic and varies with depth below the surface as a function of the overlying weight of water. Pore-water pressure can exceed or fall short of hydrostatic under hydrodynamic conditions or if a soil compacts or dilates under load (e.g., Iverson et al., 2000; Major, 2000). Below the water table, soil compaction will cause a transient increase in pore-water pressure, the duration and magnitude of which are governed mainly by the rate of compaction and the permeability of the soil. An increase in pore-water pressure can lead to a loss of soil strength. If compaction thoroughly disrupts intergranular contacts, then the pore fluid may bear the entire weight of the solid grains, and the soil will liquefy. In unsaturated soil, such as occurs above the water table or (occasionally) when a saturated soil dilates (expands) under load, water does not fill pore space completely, and pore-water pressure locally is less than atmospheric. In that case, capillary and electrostatic forces cause water to adhere to solid particles. As soil water content decreases, tensional force increases and negative pore-water pressure (also known as suction) bonds solid particles, increasing the soil strength. The magnitude of negative pore-water pressure depends on soil texture and physical properties, as well as on water content. Fine-grained soils have a broader pore-size distribution and larger particle-surface area than do coarse soils. As a result, fine-grained soils have a greater range of negative-pressure potential because they can hold more water than coarse soils and because the water bonds more tightly to particle surfaces. Piezometers are used to measure positive pore-water pressure; tensiometers commonly are used to measure negative pore-water pressure (e.g., Reeve, 1986).

### 7.3.4.4 Effective Stress

The behavior of porous media having fluid-filled pores depends not only on the total state of stress to which the material is subjected, but also on the pressure of the pore fluid. The state of stress that causes solid-body deformation is the stress that acts on the skeleton of solid material that makes up the porous medium; however, that stress is modulated by the pressure of the pore fluid. Therefore, when dealing with porous media the total state of stress is commonly partitioned into components that describe the fluid pressure and the stress acting on the solid skeleton. Such partitioning of stress leads to the concept of effective stress, a concept partly recognized by Charles Lyell as early as the late 1800s (Skempton, 1960), but not explicitly articulated until Terzaghi (1923, 1943) proposed a simple theoretical framework for soil consolidation. The concept of effective stress is elegantly simple and is defined as the difference between total stress and pore-fluid pressure. Effective stress should not be confused with deviatoric stress, which is the difference between the total stress and mechanical mean pressure exerted on a homogenous, nonporous medium.

The mathematical formulation most useful for describing effective stresses in soils and other compressible porous media
Original position of surface

Hydrostatic fluid pressure

Total principal stress (Total fluid pressure)

Hydrostatic fluid pressure

Total principal stress

Hydrostatic fluid pressure

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is given by (e.g., Lambe and Whitman, 1969; Iverson and Reid, 1992; Middleton and Wilcock, 1994)

\[ \sigma'_{ij} = \sigma_{ij} + p \delta_{ij} \]  

where \( \sigma'_{ij} \) are the effective stresses acting on the solid skeleton, \( \sigma_{ij} \) the total stresses acting on the porous medium, and \( p \) the pore-fluid pressure. Note that the effective stress in this formulation appears to be an additive function of total stress and pore-fluid pressure, but recall that normal stresses are defined negative in compression whereas pore-fluid pressure is defined positive in compression.

Partitioning the total stress in terms of effective stress and fluid pressure illuminates crucial physical insights. Consider a saturated porous medium in which water statically fills the pores. If that saturated medium is now submerged beneath a water surface, both the total stress exerted on the medium and the pore-water pressure increase by an equal amount. As a result, the effective stress remains unchanged. Hence, simply increasing the fluid pressure does not cause a volume change of the medium. Now consider a container of laterally confined saturated porous material, let's say saturated sediment. If a vertical load is added instantaneously to the sediment surface across an impermeable barrier that prevents pore-water drainage, the total stress within the sedimentary body increases. In response to that stress change, the sediment grains attempt to pack closer together. However, because the pore water cannot escape and because we shall assume that both the water and the sediment grains are incompressible, particle rearrangement cannot occur. As a result, the intergranular stresses acting on the sediment grains cannot change, the sedimentary body cannot compact, and the water pressure increases by an amount equal to the change in total stress. Again, we find that simply increasing pore-water pressure does not cause volume change. Now consider the case in which the vertical load is applied across a drainage panel atop the sediment body, which allows pore water to drain. Because the water pressure within the sedimentary body has increased above hydrostatic in this example, pore water flows toward the drainage panel at the deposit surface (where the water pressure is zero) in response to the change in gradient of the hydraulic head caused by the change in pore-water pressure. As pore-water seepage progresses, the pressure in excess of hydrostatic is gradually diminished and transferred to the stress acting on the sediment grains and the deposit compacts. Compaction, or volume change, therefore, occurred in response to changes in the intergranular, or effective, stress. Thus, in porous media, the measureable effects from changes in stress, such as volume change, distortion, or changes in shear resistance, are due exclusively to changes in effective stress (Terzaghi, as quoted in Skempton, 1960).

We can solidify the thought experiment in more concrete terms by examining fluid pressure and total stress within a shallow, one-dimensional, water-saturated sediment deposit in which the vertical coordinate direction, \( y \), is defined positive upward (Figure 2). If water statically fills the pores within the sediment body, then the hydrostatic fluid pressure, \( p_w \), of a column of water extending from the surface to a depth \( H - y \) is

\[ p_w = \rho_w g (H - y) \]  

where \( \rho_w \) is the density of water, \( g \) the gravitational acceleration in the coordinate direction, and \( H \) the coordinate value identifying the body surface (e.g., Major, 2000). The total stress acting on the sediment body, extending from the surface to the same depth, is

\[ \sigma_{yy} = - \rho_w g (H - y) \]  

where \( \rho_s \) is the total mass density of the water-saturated sediment. (The negative sign follows the convention that total stress is defined as negative in compression, whereas pore-fluid pressure is defined as positive in compression.) The total mass density for the body can be written in terms of water density, \( \rho_w \) grain density, \( \rho_s \) and porosity, \( n \) (assumed here to be uniform throughout the depth of the shallow body), as

\[ \rho_s = \rho_w n + \rho_s (1 - n) \]  

Substitution of eqn [21] into eqn [20], and some algebraic manipulation, leads to

\[ \sigma_{yy} = - [\rho_w + (\rho_s - \rho_w)(1 - n)] g (H - y) \]  

This expression shows that the total stress at depth in a column of uniformly porous, water-saturated sediment depends on the weight of the overlying water plus the buoyant weight of the column of overlying solids (e.g., Major, 2000).

Suppose now the saturated sediment is loaded rapidly but with no change of stress at the deposit surface caused (e.g., by rapid deposition of a uniform thickness of similar saturated sediment). As a result, the pore-water pressure changes because the water that fills the pores is incompressible and it...
resists particle rearrangement. That resistance leads to a temporary increase in fluid pressure. The total pore-water pressure can then be written as $p_t = p_h + p'$, where $p_h$ represents the hydrostatic portion of the pressure and $p'$ represents the water pressure that is in excess of hydrostatic. Under rapid loading, water does not drain instantaneously from the pores; instead, it temporarily bears the weight of the new load. If the water bears the entire total stress imposed on the system, the sediment is said to be liquefied. Setting the total water pressure equal to the total stress in eqn [22] and recasting the expression leads to

$$p_t = p_h g(H - y) + (p_e - p_w)(1 - n)g(H - y) \tag{23}$$

The first term on the right-hand side of this expression is the hydrostatic pressure and the second term is the nonequilibrium, or excess, water pressure. This expression shows that the excess water pressure is equal to the buoyant unit weight of the sediment (Figure 2(a)). Owing to the head gradient that is established because of the nonequilibrium water pressure, the pore water will flow down gradient, from high head to low head. As it does, the excess water pressure will diffuse and the effective stress acting on the solid skeleton will increase (Figure 2(b)). At infinite time, all of the excess water pressure will have dissipated, the sediment deposit will have consolidated, and the effective stress will equal the difference between the total stress acting on the system and the hydrostatic water pressure – an equilibrium state in which no further volume change can occur.

### 7.3.5 Deformation

Body forces and stresses cause both solids and fluids to deform. In general, solids will distort only a certain amount upon application of a stress or will not distort at all until the applied stress exceeds a certain value. Fluid on the other hand – here restricted to liquid – typically will distort immediately and continuously upon application of a stress; some special classes of liquids, however, will distort continuously only after an applied shear stress exceeds a certain value. These differences in the effects that forces and stresses produce in solids and liquids lead to the concepts of deformation, strain, and rate of deformation.

Solid-body deformation can be described in terms of a displacement field, an array of displacement vectors that connect the positions of the elements of a body before and after deformation (Means, 1976; Middleton and Wilcock, 1994; Turcotte and Schubert, 2002). Under this description, deformation includes rigid-body translation, rigid-body rotation, and distortion (strain). For example, if a displacement vector translates or rotates but does not change length, then it undergoes a rigid-body deformation but not a strain. Strain requires a distortion – a change in length of a displacement vector or a change in angle between perpendicular elements of the displacement vectors. Changes in length are related to elongation or contraction of a displacement vector and are referred to as normal strain. Angular changes between perpendicular elements of a displacement vector are referred to as shear strain. Description of deformation in this manner differs from its description in the engineering literature. Bird et al. (1960) and Malvern (1969), for example, treated deformation as synonymous with strain or distortion and all rigid-body motion as rigid-body displacement, not deformation.

#### 7.3.5.1 Normal Strain

We begin our consideration of normal strain by examining the $x$-direction displacement of a line having endpoints $a$ and $b$ and length $\delta x$ (Figure 3). The $x$-direction coordinates of $a$ and $b$ are $x$ and $x + \delta x$, respectively. We now suppose the line is displaced to a new position having endpoints $a'$ and $b'$ and length $\delta x'$. The respective $x$-direction coordinates of the endpoints of the displaced line are $x'$ and $x' + \delta x'$ (Figure 3).

Displacement of point $a$ in the $x$-direction is given by $(U_a)_b = x' - x$. Displacement of point $b$ in the $x$-direction is given by $(U_b)_b = (x' + \delta x') - (x + \delta x)$. If $(U_a)_b$ is equal to $(U_b)_b$ then the line has been translated rigidly in the $x$-direction but not strained (i.e., $\delta x' = \delta x$). If the $x$-direction displacements of points $a$ and $b$ are unequal, then the line has been strained in the coordinate direction.

By using Taylor’s theorem (Zwillinger, 2003), we can represent the $x$-direction displacement of point $b$ as

$$(U_b)_b = (U_a)_0 + \frac{\partial U_a}{\partial x} \delta x + \frac{\partial^2 U_a}{\partial x^2} \left(\frac{\delta x^2}{2}\right) + \text{higher order terms} \tag{24}$$

This theorem states that the displacement of point $b$ can be represented as the displacement of point $a$ plus the product of the gradient of displacement in the $x$-direction and the original line length plus an expansion series of higher-order terms involving products of derivatives of displacement and powers of original line length. Substituting this expression and the definition of $(U_a)_b$ into the expression defining $(U_b)_b$ allows us to write an expression for the $x$-direction displacement of point $b$ as

$$(x' + \delta x') - (x + \delta x) = (x' - x) + \frac{\partial U_a}{\partial x} \delta x + \frac{\partial^2 U_a}{\partial x^2} \left(\frac{\delta x^2}{2}\right) + \text{higher order terms} \tag{25}$$
which can be recast as
\[
\left( \frac{\delta x - \delta x}{\delta x} \right) = \frac{\partial U_x}{\partial x} + \frac{\partial^2 U_x}{\partial x^2} \left( \frac{\delta x}{2} \right) + \text{higher order terms} \tag{26}
\]

By definition, the normal component of strain is related to a change in line length as
\[
\varepsilon = \frac{\Delta l}{l} \tag{27}
\]

where \( \Delta l \) is the change of line length and \( l \) the initial, undeformed line length. Strain is, therefore, a dimensionless quantity. We can relate the definition of normal strain to the one-dimensional change of line length in the \( x \)-direction by combining eqns [26] and [27] as
\[
\varepsilon_{xx} = \frac{\delta x - \delta x}{\delta x} = \frac{\partial U_x}{\partial x} + \frac{\partial^2 U_x}{\partial x^2} \left( \frac{\delta x}{2} \right) + \text{higher order terms} \tag{28}
\]

A simple characterization of the state of strain is obtained for infinitesimal strains, those that are so small that only linear relationships need be considered. For such small strains, we can neglect all higher-order terms of the expansion series. In practice, infinitesimal strain theory gives good results for strains as large as a few tenths of a percent to perhaps \( \sim 1\% \) (Malvern, 1969; Middleton and Wilcock, 1994). Consideration of infinitesimal strain is useful even in a body undergoing large strain, because an infinitesimal strain over an infinitesimal increment of time can be related to an instantaneous stress and provide useful insight (Middleton and Wilcock, 1994). Neglecting higher-order terms and taking the limit as \( \delta x \) goes to zero, eqn [28] becomes
\[
\varepsilon_{xx} = \frac{\partial U_x}{\partial x} \tag{29}
\]

Infinitesimal normal strain, defined as positive in elongation, is, thus, related to the gradient of displacement along a coordinate direction. In a similar manner,
\[
\varepsilon_{yy} = \frac{\partial U_y}{\partial y} \tag{30}
\]
\[
\varepsilon_{zz} = \frac{\partial U_z}{\partial z}
\]

With regard to strain identification, the axes of strain (and, thus, the planes of strain) are indicated by the subscripts. Repeated subscripts indicate normal strain along a coordinate axis; unequal subscripts indicate shear strain within the plane defined by the subscript axes.

Let us now consider a small elemental area having sides of length \( \delta x \) and \( \delta y \), respectively. Let each of those sides be elongated by an amount \( \delta U_x \) and \( \delta U_y \) but remain perpendicular (Figure 4). We now define the fractional change of the area of the element as
\[
\xi_A = \frac{\Delta A}{A} \tag{31}
\]

\[
\delta U_y = (\partial U_y/\partial y) \delta y
\]
\[
\delta U_x = (\partial U_x/\partial x) \delta x
\]

where \( A \) is the original area of the element. If we again use Taylor's theorem to represent the displacements of the endpoints of the respective vector components of the elemental area and assume infinitesimal strain in order to neglect higher order terms of the series expansion, we can define the new area of the deformed element as
\[
A_1 = \left( \delta x + \delta U_x \right) \left( \delta y + \delta U_y \right)
\]

Substituting the last expression of eqn [32] into eqn [31] and taking the limit as \( \delta x \) and \( \delta y \) go to zero yields
\[
\frac{\Delta A}{A} = \frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}
\]
\[
\xi_A = \frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} = \varepsilon_{xx} + \varepsilon_{yy} \tag{34}
\]

For infinitesimal strain, \( \partial U_x/\partial x \) and \( \partial U_y/\partial y \) are much less than 1 and, thus, the product term is very small and can be neglected. In two dimensions, the areal strain is thus given by
\[
\xi_A = \frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} = \varepsilon_{xx} + \varepsilon_{yy}
\]

For large strains, product and higher-order terms may not be small, and at least some of them must be retained and evaluated. Extending this infinitesimal strain analysis to three dimensions shows that the volume strain, or dilatation, of an element is given by
\[
\xi = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}
\]

Because normal strains are defined as positive in elongation, dilatation is positive. Compression, the opposite of dilation, is negative.
7.3.5.2 Shear Strain

Shear strain is measured as a change in angle between lines that were originally perpendicular. Consider an elemental area that undergoes a distortion that produces angular changes, but which leaves the sides of the area approximately the same length (Figure 5). The angular change along the x-axis, \(a_1\), is defined as positive in an anticlockwise direction because it produces a displacement along the positive y-axis. Likewise, the angular change along the y-axis, \(a_2\), is defined as positive in a clockwise direction because it produces a displacement along the positive x-axis. This produces a condition known as pure shear, provided that \(a_1 = a_2\). From geometric considerations, we can define

\[
\tan a_1 = \frac{\delta U_y}{\delta x} \\
\tan a_2 = \frac{\delta U_x}{\delta y}
\]

For very small angles, those that are small compared to one radian, \(\tan \approx z\), and therefore

\[
a_1 \approx \frac{\delta U_y}{\delta x} \\
a_2 \approx \frac{\delta U_x}{\delta y}
\]

We now define the average angular change from the original right angle of the elemental area, or average shear strain, as

\[
e_{xy} = \frac{1}{2}(a_1 + a_2)
\]

which upon substitution and taking the limit as \(\delta x\) and \(\delta y\) go to zero yields

\[
e_{xy} = \frac{1}{2} \left( \frac{\partial U_y}{\partial x} + \frac{\partial U_x}{\partial y} \right) = \frac{1}{2} \left( \frac{\partial U_y}{\partial y} + \frac{\partial U_x}{\partial x} \right) = e_{yx} \quad [39]
\]

We see from this definition that \(e_{xy} = e_{yx}\); hence, the components of shear strain are symmetric.

Similar derivations for \(e_{xz}\) and \(e_{yz}\) lead to

\[
e_{xz} = \frac{1}{2} \left( \frac{\partial U_z}{\partial x} + \frac{\partial U_x}{\partial z} \right) \\
e_{yz} = \frac{1}{2} \left( \frac{\partial U_z}{\partial y} + \frac{\partial U_y}{\partial z} \right)
\]

The total shear strain, or engineering component of the shear strain, is the sum of the angular changes \(a_1\) and \(a_2\), and is twice \(e_{xy}\). Most geological analyses of mechanics of materials utilize the definition of average shear strain.

We can now combine the definitions for normal strain and average shear strain into a strain tensor as

\[
e_{ij} = \left( \begin{array}{ccc}
\frac{\partial U_x}{\partial x} & \frac{1}{2} \left( \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial x} \right) & \frac{1}{2} \left( \frac{\partial U_z}{\partial z} + \frac{\partial U_x}{\partial y} \right) \\
\frac{1}{2} \left( \frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x} \right) & \frac{\partial U_y}{\partial y} & \frac{1}{2} \left( \frac{\partial U_z}{\partial z} + \frac{\partial U_y}{\partial x} \right) \\
\frac{1}{2} \left( \frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x} \right) & \frac{1}{2} \left( \frac{\partial U_z}{\partial y} + \frac{\partial U_y}{\partial z} \right) & \frac{\partial U_z}{\partial z}
\end{array} \right)
\]

7.3.5.3 Rotation

Angular deformation leads to the condition of pure shear if \(a_1\) is equal to \(a_2\). Angular deformation of an element can also lead to solid-body rotation, which can occur without distortion or with distortion if \(a_1\) is unequal to \(a_2\). To examine displacement relations in terms solely of rotation, consider an elemental area that undergoes an anticlockwise rotation (Figure 6), which we shall define as a positive rotation. In this

Figure 5  Definition sketch of shear strain of an elemental area under a condition of pure shear (the element is elongated in one direction and shortened at right angles to that direction). Although the angle between originally perpendicular line segments has changed, under infinitesimal strain the lengths of the strained and unstrained line segments remain similar and the angular change is very small compared to one radian. The dashed line shows the original shape of the element.

Figure 6  Definition sketch of a rotated, but unstrained, elemental area. The rotation angle is small compared to one radian.
case, displacement is positive in the y-direction and negative in the x-direction, and we shall use a subscript naming convention \((\omega_i)\) to indicate that the rotation is in the positive y-direction and negative x-direction of the xy-plane. From geometrical arguments, we can show that

\[
\sin z_1 = \frac{\delta U_y}{\delta x}, \quad \sin z_2 = -\frac{\delta U_x}{\delta y}
\]

For very small angles, \(\sin z \approx z\), and therefore \(z_1 \approx \delta U_y/\delta x\) and \(z_2 \approx -\delta U_x/\delta y\). We now define the average rotational deformation in a positive sense as

\[
\omega_{xy} = \frac{1}{2}(z_1 + z_2)
\]

Substituting the definition of \(z_1\) and \(z_2\) into eqn [42] and taking the limit as \(\delta x\) and \(\delta y\) go to zero shows that positive rotation in the xy-plane is given by

\[
\omega_{xy} = \frac{1}{2}\left(\frac{\partial U_y}{\partial y} - \frac{\partial U_x}{\partial x}\right)
\]

Unlike the components of shear strain, the components of rotation are not symmetric; instead, \(\omega_{yx} = -\omega_{xy}\). Similar derivations lead to

\[
\omega_{yx} = \frac{1}{2}\left(\frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y}\right)
\]

\[
\omega_{xz} = \frac{1}{2}\left(\frac{\partial U_z}{\partial x} - \frac{\partial U_x}{\partial z}\right)
\]

Combining all the components of rotation into a rotation tensor yields

\[
\omega_{ij} = \begin{pmatrix}
0 & \frac{1}{2}\left(\frac{\partial U_y}{\partial y} - \frac{\partial U_x}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial U_y}{\partial z} - \frac{\partial U_z}{\partial y}\right) \\
\frac{1}{2}\left(\frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial U_z}{\partial x} - \frac{\partial U_x}{\partial z}\right) \\
\frac{1}{2}\left(\frac{\partial U_y}{\partial z} - \frac{\partial U_z}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial U_z}{\partial y} - \frac{\partial U_y}{\partial z}\right) & 0
\end{pmatrix}
\]

If we now sum the strain tensor (eqn [41]) and the rotation tensor (eqn [45]), we have a deformation tensor that defines simple displacement gradients:

\[
\frac{\partial U_i}{\partial x_j} = \begin{pmatrix}
\frac{\partial U_x}{\partial x} & \frac{\partial U_x}{\partial y} & \frac{\partial U_x}{\partial z} \\
\frac{\partial U_y}{\partial x} & \frac{\partial U_y}{\partial y} & \frac{\partial U_y}{\partial z} \\
\frac{\partial U_z}{\partial x} & \frac{\partial U_z}{\partial y} & \frac{\partial U_z}{\partial z}
\end{pmatrix}
\]

From this analysis, we see that we cannot define distortion, or strain, solely in terms of simple displacement gradients. To do so is to mix both rotation and shear strain in the off-diagonal terms (Middleton and Wilcock, 1994). Because solid-body translations and rotations do not alter distances between neighboring elements of a solid, expressions that link stresses to strains concern only the strain tensor, which is sufficient to describe infinitesimal strains occurring in solid bodies subject to stresses (Turcotte and Schubert, 2002).

### 7.3.5.4 Strains in an Arbitrarily Oriented Coordinate System

As with stresses, having expressions for the changes in lengths and rotation angles of line elements that are oriented in any coordinate direction is useful. Defining strain in an arbitrary coordinate direction also involves resolving trigonometric relationships. Similar to the discussion of stresses, the discussion for strains focuses on two dimensions and is relative to strains defined as positive in elongation. Strains within an xy-plane referenced to a new coordinate system that makes an angle \(\theta\) with the original xy-coordinate system can be written as (Turcotte and Schubert, 2002; Middleton and Wilcock, 1994)

\[
e_{xx'} = e_{xx}\cos^2\theta + e_{yy}\sin^2\theta + e_{xy}\sin 2\theta
\]

\[
e_{yy'} = e_{xx}\sin^2\theta + e_{yy}\cos^2\theta - e_{xy}\sin 2\theta
\]

\[
e_{xy'} = \frac{1}{2}(e_{xx} - e_{yy})\sin 2\theta - e_{xy}\cos 2\theta
\]

Similar to the principal axes that were defined for stresses, principal axes of strain can be defined along directions in which shear strain is zero by setting \(e_{xy'}\) equal to zero in eqn [49], which gives

\[
\tan 2\theta = \frac{2e_{xy}}{e_{xx} - e_{yy}}
\]

Again, \(\theta\) and \(\theta + \pi/2\) define the directions of the principal strains \(e_1\) and \(e_2\). By setting \(e_{yy'} = 0\), \(e_{xx'} = e_1\), and \(e_{xy'} = e_2\), we can write expressions for the normal strains and shear strains in a coordinate system that makes an angle \(\theta\) with respect to the principal strain \((e_1)\) axis as (Turcotte and Schubert, 2002; Middleton and Wilcock, 1994)

\[
e_{xx} = e_1\cos^2\theta + e_2\sin^2\theta
\]

\[
e_{yy} = e_1\sin^2\theta + e_2\cos^2\theta
\]

\[
e_{xy} = \frac{1}{2}(e_1 - e_2)\sin 2\theta
\]

In eqns [51]–[53], the prime notation has been dropped as it merely indicates the components in a new coordinate system. Tensor notation expressions for three-dimensional infinitesimal strains of lines rotated with respect to a defined coordinate system are discussed in Means (1976).

### 7.3.6 Rate of Deformation

The discussion of solid-body deformation considered the kinematic behavior (analysis of motions without reference to the forces involved) of a solid undergoing a very small geometric change. Because the strains and rotations that defined deformation were very small we focused on displacements, displacement fields, and displacement gradients relative to material coordinates in an undeformed reference state, a state
very similar to the final deformed state. When considering fluid deformation, however, displacements of fluid elements are not small, but instead are very large or indefinite. For example, if one poured maple syrup on a plate and tilted the plate, the syrup would flow as long as the plate remained tilted. Therefore, in fluids, we are concerned not with displacements, but rather with rates of displacements, or velocities. Instead of considering deformation, we consider instead rates of deformation and focus on velocities, velocity fields, and velocity gradients. These fields are expressed in terms of spatial coordinates and time without regard to an initial state. Similar to the discussion of strain and deformation, rate of deformation in a broader sense than that typically used in the engineering literature (e.g., Malvern, 1969) is used. Here, rate of deformation encompasses both rate of distortion (commonly called rate of strain), which involves what we might think of as rate of dilatation and rate of shear, and which Malvern (1969) calls rate of deformation, and rate of 'rigid-fluid' rotation, called vorticity, which Malvern (1969) calls spin.

It is tempting to relate the rate-of-deformation tensor to the time derivative of the deformation tensor (eqn [41]), but this is not the general case (Malvern, 1969). As noted, deformation is defined in terms of material coordinates relative to an initial reference state, whereas the rate of deformation is defined with respect to the present spatial coordinates of a particle. Consider an observer recording fluid velocities at some point in a system. That observer cannot see the initial position of, or the trajectory followed by, a fluid particle, but can only record the particle velocity as a function of spatial position and time. The components of the rate-of-deformation tensor are, therefore, founded in terms of velocity of a particle instantaneously at one point relative to a particle instantaneously at a neighboring position (Malvern, 1969), or spatial gradients of instantaneous velocity. The rate-of-deformation tensor and the time derivative of the deformation tensor are equivalent only when the displacements and displacement gradients are very small. For fluid flow, however, displacements of fluid particles are not small. Because rate of deformation involves spatial gradients of velocity, it has a unit of s⁻¹.

Like the deformation tensor, the rate-of-deformation tensor can be partitioned into a symmetric tensor component that represents the rate of distortion of a fluid element, and an antisymmetric tensor component that represents a 'rigid-body' rate of rotation, or vorticity (Malvern, 1969; Tritton, 1988; Middleton and Wilcock, 1994). Each tensor is comprised of components that describe spatial gradients of velocity. The rate-of-distortion tensor (commonly called the rate-of-strain tensor in geosciences) is given by

\[
\Lambda_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} \right) + \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} \right) - \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} \right) \left( \frac{\partial u_i}{\partial x_j} \right)
\]

in which \( u_i \) represents the velocity in the Cartesian coordinate direction \( i \). The similarity of the rate-of-distortion tensor and the strain (or distortion) tensor is apparent. By analogy with solid-body strain, a fluid dilatation rate (rate of volume change) can be defined by the diagonal terms of the rate-of-distortion tensor as

\[
D = \Lambda_{xx} + \Lambda_{yy} + \Lambda_{zz} = \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right)
\]

We shall see subsequently that this sum of spatial gradients of velocity along coordinate directions, also known as the divergence of the velocity field, is an important component of the conservation of mass equation. The off-diagonal terms of the rate-of-distortion tensor define rates of distortion related to shear.

In an analogous manner, the rate-of-rotation, or vorticity, tensor is given by

\[
\Omega_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)
\]

As with strain and rotation, the components of the rate-of-distortion and vorticity tensors can be combined into a rate-of-deformation tensor that defines rates of dilatation, rates of shear, and rates of rotation. As with deformation, we cannot define rates of distortion, the velocity gradients responsible for distorting the shape of a fluid element, solely in terms of simple velocity gradients because to do so is to mix both rates of shear and rates of rotation in the off-diagonal terms.

### 7.3.7 Conservation

Having examined the concepts of stress, deformation, and rates of deformation, the chapter turns to the concept of conservation equations, which are sometimes called equations of change because they describe changes in quantities such as temperature, density, and velocity with respect to time and space (Bird et al., 1960). Conservation is one of the key principles of continuum mechanics used for solving geomorphic problems quantitatively (Anderson and Anderson, 2010). Conservation equations typically state that the time rate of change of some quantity within a definable volume equals the amount of the quantity transported into the volume plus the amount of that quantity produced (or consumed) within the volume minus the amount of that quantity transported out of the volume. Conservation of physical quantities is nature’s equivalent of balancing a budget. For problems in geomorphology, we are concerned particularly with conservation of mass and momentum, but some problems also rely on conservation of energy and conservation of chemical species (a form of mass conservation) (e.g., Anderson and Anderson, 2010).
7.3.7.1 Conservation of Mass

For a stationary elemental volume (one of fixed size having a position fixed in space), the rate of mass accumulating within the volume is equal to the mass flux into the volume minus the mass flux out of the volume. We shall assume that no mass is generated (or consumed) within the volume (i.e., we ignore relativistic effects); hence, the only way to have mass accumulate in the volume is to have it flow across the volume boundary. By definition, material density is the amount of mass per unit volume; thus, we can represent mass as the product of density and volume. Consider the pair of faces of the elemental volume perpendicular to the x-axis, as shown in Figure 7. The flow of mass entering the volume across the left-hand face (at \( x \)) is given by the product of material density, the area of the face across which the mass flows, and the velocity at which the mass flows \((\rho u_x \delta y \delta z)\). Likewise, the flow of mass leaving the volume across the right-hand face (at \( x + \delta x \)) is given by \((\rho u_x \delta y \delta z)\). Similar expressions can be written for the mass fluxes across faces perpendicular to the y and z axes. Using the expressions for mass flux across all faces of the elemental volume, we can write the principle of mass conservation in mathematical terms as

\[
\frac{\partial}{\partial t} (\rho \delta x \delta y \delta z) = [(\rho u_x \delta y \delta z)_x - (\rho u_x \delta y \delta z)_{x+\delta x}]
\]

\[
+ [(\rho u_y \delta x \delta z)_y - (\rho u_y \delta x \delta z)_{y+\delta y}]
\]

\[
+ [(\rho u_z \delta x \delta y)_z - (\rho u_z \delta x \delta y)_{z+\delta z}] \quad [57]
\]

Because the volume is stationary, we can recast eqn [57] as

\[
\frac{\partial}{\partial t} (\rho \delta x \delta y \delta z) = \delta x \delta y \delta z [(\rho u_x)_x - (\rho u_x)_{x+\delta x}]
\]

\[
+ \delta x \delta y [(\rho u_y)_y - (\rho u_y)_{y+\delta y}]
\]

\[
+ \delta x \delta y [(\rho u_z)_z - (\rho u_z)_{z+\delta z}] \quad [58]
\]

Dividing eqn [58] by \( \delta x \delta y \delta z \) and taking the limit as these dimensions go to zero yields:

\[
\frac{\partial \rho}{\partial t} = -\left( \frac{\partial \rho u_x}{\partial x} + \frac{\partial \rho u_y}{\partial y} + \frac{\partial \rho u_z}{\partial z} \right) \quad [59]
\]

This equation, also known as the continuity equation, states that the rate of change of density within an elemental volume fixed in space is equal to the net rate of mass flux across its boundaries divided by its volume (Bird et al., 1960). The negative sign in front of the right-hand side of the equation indicates that if the net gradient of mass flux decreases along the coordinate directions within the volume, then the change of density with time is positive (because mass accumulates). If, however, the net gradient of mass flux increases along the coordinate directions within the volume, the change of density with time is negative (because mass is lost).

The derivation of eqn [59] assumed that the elemental volume was fixed in space. Suppose, however, that we allow the volume to float along with the medium that is transporting mass across the volume boundary. What happens to our expression for rate of change of density? Insight to that question can be gleaned if we carry out the differentiation of the term on the right-hand side of eqn [59] and regroup terms. Doing so yields

\[
\frac{\partial \rho}{\partial t} + u_x \frac{\partial \rho}{\partial x} + u_y \frac{\partial \rho}{\partial y} + u_z \frac{\partial \rho}{\partial z} = -\rho \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) \quad [60]
\]

The term on the left-hand side of eqn [60] now describes rate changes of density detected from the perspective of an observer floating along with the motion of the medium that is transporting the mass. The first term describes the rate of change observed from general variations with time, and the following three terms describe an additional change related to any spatial gradients of density. The rate of change observed from spatial gradients of density is related to the local velocity of the transporting medium and is known as the convective rate of change. The term on the left-hand side of eqn [60] represents a special kind of time derivative called the 'substantial' or 'material' derivative, or the time derivative following motion (Bird et al., 1960). In this instance, that special derivative is expressed as

\[
\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u_x \frac{\partial \rho}{\partial x} + u_y \frac{\partial \rho}{\partial y} + u_z \frac{\partial \rho}{\partial z} \quad [61]
\]

Using that special derivative notation, and with slight rearrangement, we can write eqn [60] as

\[
\frac{D\rho}{Dt} + \rho \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) = 0 \quad [62]
\]

For an incompressible medium (commonly a valid approximation for many geomorphic problems), density is constant. Thus, \( D\rho/Dt = 0 \), and, therefore, the sum of the spatial gradients of the velocity field in the second term (known as the divergence of the velocity field) must also be zero. Equations [59] and [62] are commonly written in a more compact
notation, using what is called the divergence operator, as

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{u} \quad \text{and} \quad \frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u}
\]  

[63]

For an incompressible material \( \nabla \cdot \mathbf{u} = 0 \).

### 7.3.7.2 Conservation of Linear Momentum

A very powerful concept in continuum mechanics is the principle of conservation of momentum. This concept states that the time rate of change of momentum in an elemental control volume equals the flow of momentum into the control volume minus the flow of momentum leaving the volume plus the sum of the forces acting on the volume. This is essentially a control volume formulation of Newton's second law that accounts not only for the forces acting on the volume, but also for the impact of momentum flowing across the control volume boundary (Bejan, 1984). Let us again consider a stationary elemental volume depicted in Figure 8(a), and examine the x-direction component of the conservation of momentum expression.

Momentum is the product of mass and velocity. From the analysis of mass conservation, we saw that mass is represented as the product of density and volume. We can, thus, express momentum as the product of density, volume, and velocity. Therefore, the flow of momentum entering the volume across the left-hand face (at \( x \)) is given by \((\rho u_x, \rho u_y, \rho u_z)\), and the flow of momentum leaving the volume across the right-hand face (at \( x + \Delta x \)) is given by \((\rho u_x', \rho u_y', \rho u_z')\). Note, however, that x-direction components of momentum are also crossing other faces of the volume. The x-direction component of momentum entering the volume across its bottom face (at \( y \)) is given by \((\rho u_x, \rho u_y, \rho u_z)\), and the x-direction momentum flowing out the top face (at \( y + \Delta y \)) is given by \((\rho u_x, \rho u_y', \rho u_z')\). Similar expressions can be written for the x-direction component of momentum flowing across the rear and front faces of the volume. Gathering these terms, we can write an expression for the net convective flux of the x-direction component of momentum across the volume boundary:

\[
\begin{align*}
(\rho u_x, \rho u_y, \rho u_z)_{x+\Delta x} &- (\rho u_x, \rho u_y, \rho u_z)_{x-\Delta x} + (\rho u_x', \rho u_y', \rho u_z')_{x+\Delta x} - (\rho u_x', \rho u_y', \rho u_z')_{x-\Delta x} \\
&- (\rho u_x, \rho u_y, \rho u_z)_{y+\Delta y} + (\rho u_x, \rho u_y', \rho u_z')_{y+\Delta y} - (\rho u_x, \rho u_y', \rho u_z')_{y-\Delta y} \\
&- (\rho u_x', \rho u_y, \rho u_z')_{z+\Delta z} + (\rho u_x', \rho u_y, \rho u_z')_{z-\Delta z} - (\rho u_x', \rho u_y, \rho u_z')_{z+\Delta z} - (\rho u_x', \rho u_y, \rho u_z')_{z-\Delta z} \\
&- (\rho u_x, \rho u_y, \rho u_z)_{x+\Delta x} + (\rho u_x', \rho u_y', \rho u_z')_{x+\Delta x} - (\rho u_x', \rho u_y', \rho u_z')_{x-\Delta x} \\
&- (\rho u_x, \rho u_y, \rho u_z)_{y+\Delta y} + (\rho u_x', \rho u_y', \rho u_z')_{y+\Delta y} - (\rho u_x', \rho u_y', \rho u_z')_{y-\Delta y} \\
&- (\rho u_x', \rho u_y, \rho u_z')_{z+\Delta z} + (\rho u_x', \rho u_y, \rho u_z')_{z+\Delta z} - (\rho u_x', \rho u_y, \rho u_z')_{z+\Delta z}
\end{align*}
\]

[64]

In addition to the convective flux of momentum across the volume boundary, the rate of change of momentum is also affected by the forces acting on the volume, of which there are two types: body forces that affect all parts of the volume equally and surface forces related to the stresses acting on the volume. Stresses acting on the volume affect not only its bulk momentum, but also they induce diffusive molecular transfer of momentum across the volume surface (Bird et al., 1960). From Figure 8(b), we see that the sum of the x-direction components of these forces can be written as

\[
\begin{align*}
-(\sigma_{xx}\delta y\delta z)_{x+\Delta x} + (\sigma_{xx}\delta y\delta z)_{x-\Delta x} + (\sigma_{xx}\delta y\delta z)_{y+\Delta y} - (\sigma_{xx}\delta y\delta z)_{y-\Delta y} \\
-(\sigma_{yy}\delta z\delta x)_{z+\Delta z} + (\sigma_{yy}\delta z\delta x)_{z-\Delta z} + (\sigma_{yy}\delta z\delta x)_{z+\Delta z} - (\sigma_{yy}\delta z\delta x)_{z-\Delta z}
\end{align*}
\]

[65]

Here, the negative terms result from positively defined forces acting in the negative x-direction.

Noting that the time rate of change of momentum in the x-direction can be written as \( \frac{\partial}{\partial t} (\rho u_x, \rho u_y, \rho u_z) \times \), and using eqns [64] and [65], we can write an expression for the x-direction component of the conservation of momentum as

\[
\begin{align*}
\frac{\partial}{\partial t} (\rho u_x, \rho u_y, \rho u_z)_{x} &- (\rho u_x, \rho u_y, \rho u_z)_{x+\Delta x} \\
&+ (\rho u_x, \rho u_y, \rho u_z)_{x-\Delta x} + (\rho u_x', \rho u_y', \rho u_z')_{x+\Delta x} - (\rho u_x', \rho u_y', \rho u_z')_{x-\Delta x} \\
&- (\rho u_x, \rho u_y, \rho u_z)_{y+\Delta y} + (\rho u_x, \rho u_y', \rho u_z')_{y+\Delta y} - (\rho u_x, \rho u_y', \rho u_z')_{y-\Delta y} \\
&- (\rho u_x', \rho u_y, \rho u_z')_{z+\Delta z} + (\rho u_x', \rho u_y, \rho u_z')_{z+\Delta z} - (\rho u_x', \rho u_y, \rho u_z')_{z+\Delta z} - (\rho u_x', \rho u_y, \rho u_z')_{z-\Delta z} \\
&- (\rho u_x, \rho u_y, \rho u_z)_{x+\Delta x} + (\rho u_x', \rho u_y', \rho u_z')_{x+\Delta x} - (\rho u_x', \rho u_y', \rho u_z')_{x-\Delta x} \\
&- (\rho u_x, \rho u_y, \rho u_z)_{y+\Delta y} + (\rho u_x', \rho u_y', \rho u_z')_{y+\Delta y} - (\rho u_x', \rho u_y', \rho u_z')_{y-\Delta y} \\
&- (\rho u_x', \rho u_y, \rho u_z')_{z+\Delta z} + (\rho u_x', \rho u_y, \rho u_z')_{z+\Delta z} - (\rho u_x', \rho u_y, \rho u_z')_{z+\Delta z}
\end{align*}
\]  

[66]
If we assume the spatial dimensions of the elemental volume are constant, then we can divide eqn [66] by \( \delta y \delta z \), take the limit as each of those dimensions go to zero, and write the \( x \)-direction component of the conservation of momentum as

\[
\frac{\partial \rho u_x}{\partial t} = -\frac{\partial}{\partial x} (\rho u_x u_x) - \frac{\partial}{\partial y} (\rho u_x u_y) - \frac{\partial}{\partial z} (\rho u_x u_z)
+ \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho g_x
\]  

[67]

Expressions for the \( y \)- and \( z \)-direction components of the conservation of momentum are derived in a similar manner and are

\[
\frac{\partial \rho u_y}{\partial t} = -\frac{\partial}{\partial x} (\rho u_x u_y) - \frac{\partial}{\partial y} (\rho u_y u_y) - \frac{\partial}{\partial z} (\rho u_y u_z)
+ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + \rho g_y
\]  

[68]

and

\[
\frac{\partial \rho u_z}{\partial t} = -\frac{\partial}{\partial x} (\rho u_x u_z) - \frac{\partial}{\partial y} (\rho u_y u_z) - \frac{\partial}{\partial z} (\rho u_z u_z)
+ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho g_z
\]  

[69]

We can write the three components of momentum conservation in a more compact manner using vector and tensor notation as

\[
\frac{\partial \rho \mathbf{u}}{\partial t} = -\nabla \rho \mathbf{u} + \nabla \cdot \mathbf{\sigma} + \rho \mathbf{g}
\]  

[70]

in which \( \mathbf{u} \) represents the velocity vector field, \( \mathbf{\sigma} \) the stress tensor (eqn [4]), and \( \mathbf{g} \) the vector field for acceleration of gravity.

An alternative form of eqn [70] is achieved if we carry out the differentiation of the term on the left-hand side and the first group of terms on the right-hand side and regroup the terms. For simplification, consider the \( x \)-direction component of the momentum conservation equation. Equation [67], after carrying out the noted differentiation, can be rearranged as

\[
\rho \left( \frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} \right)
+ u_x \left( \frac{\partial p}{\partial x} + u_y \frac{\partial p}{\partial y} + u_z \frac{\partial p}{\partial z} + \rho \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) \right)
= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho g_x
\]  

[71]

From the previously discussed definition of the material derivative, and using eqn [61], we can recast this equation as

\[
\rho \left( \frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} \right)
+ u_x \left( \frac{\partial p}{\partial x} + u_y \frac{\partial p}{\partial y} + u_z \frac{\partial p}{\partial z} + \rho \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) \right)
= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho g_x
\]  

[72]

We now recognize the bracketed term as the mass conservation, or continuity, equation (eqn [62]), and that equation is equal to zero. Thus, we can write the \( x \)-direction component of the conservation of momentum equation as

\[
\rho \frac{\partial u_x}{\partial t} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho g_x
\]  

[73]

In three dimensions, and using vector and tensor notation, we can write this form of the conservation of momentum equation as

\[
\rho \frac{\partial \mathbf{u}}{\partial t} = \nabla \mathbf{\sigma} + \rho \mathbf{g}
\]  

[74]

This expression, which explicitly relates material accelerations (left-hand term) to surface and body forces (right-hand terms), is called Cauchy’s first law.

If we now recall that total stress can be written as the sum of the mechanical mean pressure (mean normal stress) and the deviatoric stress, we can write eqn [73] in an alternative form as

\[
\rho \frac{\partial \mathbf{u}}{\partial t} = -\left( \frac{\partial \mathbf{P}}{\partial x} \right) + \left( \frac{\partial (\sigma_{xx} + \mathbf{P})}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \right) + \rho \mathbf{g}
\]  

[75]

We can extend this to three dimensions and recast eqn [74] as

\[
\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla \mathbf{P} + \nabla \cdot \mathbf{\sigma} + \rho \mathbf{g}
\]  

[76]

The reason for presenting the conservation of momentum equation on the one hand in terms of total stresses and on the other in terms of mechanical mean pressure and deviatoric stresses is that in solid mechanics we are generally interested in total stresses, whereas in fluid mechanics we are interested in dealing explicitly with the pressure that acts on a fluid element and the stresses that deviate from pressure and act to change the shape of a fluid element. Hence, eqn [74] is the form of conservation of momentum commonly used in solid mechanics, whereas eqn [76] is the form used in fluid mechanics.

### 7.3.8 Constitutive Relations

The discussion thus far has focused on generalized aspects of forces and stresses acting on bodies, the influence of pore-fluid pressure on stresses, and development of concepts of strain and conservation. None of these concepts, however, tells us precisely how a body will respond to an applied stress – in other words, precisely how stresses and strains (or rates of distortion) are related. Linking stresses with strains or rates of distortion requires a conceptualization of how a material acts. Unlike the development of the concepts of stress, strain, and conservation from fundamental physical principles of mass, motion, and geometry, conceptual expressions of material activities are idealizations rooted in empiricism. Although real materials can exhibit very complex stress–strain relationships, and complex idealizations have been proposed to model those relationships, some relatively simple idealizations approximate activities of real materials fairly well. Expressions that describe the relationships between stress and strain, or stress and rates of distortion, are called constitutive equations.
Three basic idealizations are used to describe material actions: elastic, plastic, and viscous. Real materials can be represented by these idealizations, sometimes simply and sometimes by invoking complex combinations of idealized actions. In Earth sciences, in general, material actions are commonly explored using combinations of idealized material actions (e.g., Middleton and Wilcock, 1994). Yet, despite clearly limiting assumptions, even the simplest of idealized material actions commonly lead to substantial insights.

Special cases of idealized actions include linear elasticity, ideal plasticity, and linear viscosity. A linearly elastic material strains immediately upon application of a stress, and the amount of strain is directly proportional to the stress. As long as the stress remains, the strain persists; once the stress is removed, however, all of the strain is recovered. The classical conceptualization of a linearly elastic body is a spring. In an idealized elastic material, the body does not strain until the stress equals a threshold value, known as the yield stress. After an applied stress equals the yield stress, the material strains indefinitely so long as the stress equals the threshold value. Unlike a linearly elastic material, the amount of strain is not directly proportional to the stress, nor is it recoverable upon removal of the stress. The classical conceptualization of an ideal plastic material (also known as a Saint Venant material) is a block sitting on a surface. The block will not move until the friction between it and the surface it is sitting upon is overcome. Once the friction is overcome, the block continues to move across the surface as long as the applied stress (in this case a shear stress) is sufficient to overcome the friction. A simple viscous material, typically a liquid, strains immediately and indefinitely upon application of a shear stress. The shear stress is, therefore, not proportional to strain, but rather to how fast the material is distorting. In an idealized linearly viscous material, called a Newtonian fluid, the shear stress is linearly proportional to the rate of distortion, or the velocity gradient established within the fluid. Fluids for which that relationship is nonlinear, or in which the fluid does not respond in a viscous manner until a yield stress is exceeded, are called non-Newtonian fluids. The classical conceptualization for a Newtonian viscous fluid is an oil-filled dashpot or springless shock absorber. Another idealized behavior useful to geomorphology, known as the Coulomb failure rule, does not relate stresses to strains, but rather assumes a plastic-like behavior and relates the shearing strength of a material to its shearing resistance. This empiricism is commonly used to assess the stability of masses of rock and soil and to relate shear stresses to normal stresses in some models of flowing granular material. Next, the mathematical formulations of a few special idealized materials are examined.

### 7.3.8.1 Linearly Elastic Material

#### 7.3.8.1.1 Relationships between stress and normal strain

Linear elastic theory relates stresses to strains in an idealized elastic body and is strictly applicable only to infinitesimal elastic strains. Previously, we saw that nine components fully define each tensor (or matrix) of stress and strain. For the general case, the constitutive model that relates stress to strain in an elastic body composes a tensor (or matrix) comprised of 81 (or 9 × 9) components (e.g., Malvern, 1969; Means, 1976; Middleton and Wilcock, 1994). The coefficients that relate stresses to strains are called stiffnesses, whereas the inverse coefficients that relate strains to stresses are called compliances. Fortunately, this complex array of coefficients can be simplified. We saw that for both stresses and strains, only six of the tensor components are independent, which reduces the array of stiffnesses and compliances to 36 independent components. This array of coefficients can be further simplified, and written in terms of two elastic moduli (e.g., Middleton and Wilcock, 1994), if we assume that the elastic material under consideration is isotropic, meaning that the elastic moduli are identical in all directions. We shall see that four basic moduli are used to describe the behavior of linearly elastic material, but that only two of those moduli are independent.

An ideal linearly elastic material is described in terms of Hooke’s law, which states that stress is linearly proportional to strain (Timoshenko and Goodier, 1987). As a general concept, the law can be represented for uniaxial normal stress as

$$\sigma = E\varepsilon$$  \[77\]

in which $\sigma$ represents a uniaxial normal stress (positive in tension), $\varepsilon$ represents strain (positive in elongation), and the constant of proportionality $E$ is known as Young’s modulus.

Consider the strained body shown in Figure 9. If a tensile normal stress is exerted in the $y$-coordinate direction, Hooke’s law can be written as

$$\sigma_{yy} = E\varepsilon_{yy}$$  \[78\]

Note, however, that Young’s modulus does not fully relate the normal strains occurring in the body to the applied stress. For most elastic materials, as the body elongates in the $y$-direction, it contracts and results in a negative strain in the orthogonal directions. For example, the amount of contraction

**Figure 9** Definition sketch of normal strains in a two-dimensional linearly elastic material subject to uniaxial stress. The dashed line shows original shape of the element. Normal strains are defined positive in elongation.
in the $x$-direction is proportional to the amount of extension in the $y$-direction. The ratio of contraction to extension is known as Poisson's ratio and is expressed for uniaxial stress as

$$v = -\frac{\epsilon_{xx}}{\epsilon_{yy}}$$ \[79\]

Substituting eqn [79] into eqn [78] yields

$$\epsilon_{xx} = -\frac{v}{E} \sigma_{yy}$$ \[80\]

Elongation in the $y$-direction also causes contraction in the $z$-direction. For an isotropic material, under uniaxial stress, the amount of contraction in the $z$-direction is identical to the amount of contraction in the $x$-direction. Hence, we can write a similar expression relating strain in the $z$-direction to normal stress in the $y$-direction

$$\epsilon_{zz} = -\frac{v}{E} \sigma_{yy}$$ \[81\]

For an isotropic material subject to triaxial rather than uniaxial stress, and following along similar reasoning, we can write the normal strains along the coordinate directions in terms of Young's modulus and Poisson's ratio as

$$\epsilon_{xx} = \frac{1}{E} [\sigma_{xx} - v(\sigma_{yy} + \sigma_{zz})]$$
$$\epsilon_{yy} = \frac{1}{E} [\sigma_{yy} - v(\sigma_{xx} + \sigma_{zz})]$$ \[82\]
$$\epsilon_{zz} = \frac{1}{E} [\sigma_{zz} - v(\sigma_{xx} + \sigma_{yy})]

From these simple derivations, we see that normal strains in any coordinate direction are related to normal stresses applied in all coordinate directions and not just to the normal stress applied in the direction coincident with the normal strain. Equation [82] can also be written in terms of principal stresses and principal strains

$$\epsilon_1 = \frac{1}{E} [\sigma_1 - v(\sigma_2 + \sigma_3)]$$
$$\epsilon_2 = \frac{1}{E} [\sigma_2 - v(\sigma_1 + \sigma_3)]$$ \[83\]
$$\epsilon_3 = \frac{1}{E} [\sigma_3 - v(\sigma_1 + \sigma_2)]$$

### 7.3.8.1.2 Relationships between shear stress and shear strain

Relationships among shear stress, shear strain, and elastic moduli are derived by considering a special case of plane stress that is known as pure shear (Turcotte and Schubert, 2002). For this case, $\sigma_3 = 0$, and we shall assume $\epsilon_1 = -\epsilon_2$. Furthermore, we shall consider the strain that occurs within a plane in a coordinate system that is rotated with respect to the principal stress axes (Figure 10). Under these constraints and using eqn [83], we find that

$$\epsilon_1 = \frac{(1 + v)}{E} \sigma_1$$
$$\epsilon_2 = \frac{(1 + v)}{E} \sigma_2$$ \[84\]
$$\epsilon_3 = 0$$

In the discussion of stress, we found that the maximum shear stress occurs on a plane that makes a 45° angle to the principal stress axes. Under the assumption that $\sigma_1 = -\sigma_2$, and for $\theta = 45^\circ$, we find from eqns [12]–[14] that $\sigma_{xx} = \sigma_{yy} = 0$ and that $\sigma_{xy} = \sigma_1$. From the finding that $\sigma_{xy} = \sigma_1$ and the assumption that $\sigma_1 = -\sigma_2$, we find from eqns [84] that the principal strains are related to the maximum shear stress

$$\epsilon_1 = \frac{(1 + v)}{E} \sigma_{xy}$$
$$\epsilon_2 = -\frac{(1 + v)}{E} \sigma_{xy}$$ \[85\]

Under the conditions specified, eqn [85] shows that $\epsilon_1 = -\epsilon_2$. Using this result and with $\theta = 45^\circ$, we find from eqns [51]–[53] that

$$\epsilon_{xx} = \epsilon_{yy} = 0$$
$$\epsilon_{xy} = \epsilon_1$$ \[86\]

From this result, eqn [85] yields

$$\epsilon_{xy} = \frac{(1 + v)}{E} \sigma_{xy}$$ \[87\]

Owing to symmetry of shear stresses and shear strains, $\epsilon_{xy} = \epsilon_{yx}$. By analogy, similar expressions for shear strains in the orthogonal planes are written as

$$\epsilon_{xz} = \frac{(1 + v)}{E} \sigma_{xz}$$ \[88\]
$$\epsilon_{yz} = \frac{(1 + v)}{E} \sigma_{yz}$$

Unlike the relationships between normal stresses and elastic normal strains, the elastic shear strains are related to only a single component of shear stress and not to all components of shear stress. Furthermore, by recalling that $\epsilon_{xy}$ is the
average shear strain in the $xy$ plane and is equal to $\frac{1}{2}\gamma_{xy}$ (where $\gamma_{xy}$ is the total, or engineering, shear strain in the $xy$ plane), eqn [87] can be written as

$$\sigma_{xy} = \frac{E}{2(1+\nu)} \gamma_{xy}$$

[89]

or

$$\sigma_{xy} = G \gamma_{xy}$$

[90]

where $G = E/(1+\nu)$ is the rigidity modulus, also known as the shear modulus.

### 7.3.8.1.3 Relationship between pressure and dilatation

We can now define another elastic modulus, known as the bulk modulus, which describes the relationship between mean normal stress (or mechanical mean pressure) and volume strain (or dilatation). We can write a relationship between the mean normal stress (positive in tension) and volume strain (positive in dilation) as

$$\frac{1}{3} (\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = \kappa \varepsilon$$

[91]

Substituting eqns [35] and [82] into eqn [91] gives

$$\frac{1}{3\kappa} (\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = \frac{1}{E} [\sigma_{xx} - v(\sigma_{yy} + \sigma_{zz})] + \frac{1}{E} [\sigma_{yy} - v(\sigma_{xx} + \sigma_{zz})] + \frac{1}{E} [\sigma_{zz} - v(\sigma_{yy} + \sigma_{zz})]$$

[92]

which, after some algebraic manipulation, reduces to

$$\kappa = \frac{E}{3(1-2\nu)}$$

[93]

The coefficient $\kappa$ is the bulk modulus; its reciprocal is known as the compressibility.

We have now defined four basic moduli for a linearly elastic material, which relate stresses to infinitesimal elastic strains: $E$, $\nu$, $G$, $\kappa$. The derivations show that only two of these moduli ($E$, $\nu$) are independent; the other two ($G$, $\kappa$) are interdependent and written in terms of Young's modulus and Poisson's ratio.

French engineer Gabriel Lamé (1795–1870) wrote the first book devoted to the mathematical theory of elasticity and derived a complete set of constitutive relations for an elastic solid (e.g., Middleton and Wilcock, 1994). Commonly, his general relationship is written in terms of stresses as a function of strains as

$$\sigma_{ij} = 2G \varepsilon_{ij} + \lambda \delta_{ij} \varepsilon_{kk}$$

[94]

where $i$ and $j$ represent the coordinate directions $x$, $y$, $z$; $\varepsilon_{ij}$ represents the sum of the normal strains (dilatation); and $\delta_{ij}$ the Kronecker delta, or identity matrix. The coefficients $G$ and $\lambda$ are known as Lamé's constants. The constant $G$ is defined as before and has the simple physical interpretation as the shear modulus. The constant $\lambda$ is defined as

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

[95]

Although the bulk modulus $\kappa$ can be derived in terms of $\lambda$, we see that $\lambda$ does not have a direct physical interpretation.

### 7.3.8.2 Linearly Viscous Fluid

For a Newtonian viscous fluid, stress is linearly related to the rate of distortion. Consider, for example, the simplest case in which an incompressible fluid is subject to a shear stress in the $x$-direction and responds by distortion in the $xy$-plane (Figure 11). In that case, all velocity gradients are zero except for $\partial u_x/\partial y$, and all components of shear stress are zero except for $\sigma_{yx}$. Normal stresses (i.e., pressure) acting on the fluid may not be zero, but because the fluid is assumed incompressible they do not affect fluid distortion. For the case considered, the relationship between shear stress and rate of distortion can be expressed as

$$\tau_{yx} = \mu \frac{\partial u_x}{\partial y}$$

[96]

Here, a slightly different notation for the stress ($\tau$ instead of $\sigma$) is used, the reason for which will become apparent. The constant of proportionality, $\mu$, that relates shear stress to velocity gradient is called the dynamic viscosity, which Newton referred to as the 'lack of slipperiness' between fluid layers (Barnes et al., 1989). This coefficient has a unit that is the product of stress and time, N m$^{-2}$ s, or Pa s. We can write the relationship between shear stress and velocity gradient in a more general form by noting that the rate of distortion by shear in the $xy$-plane is written as

$$\Lambda_{yx} = \frac{1}{2} \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)$$

[97]

For the case under consideration $\partial u_x/\partial x$ is zero, and as a result we can write the rate of distortion under shear as

$$\Lambda_{yx} = \frac{1}{2} \frac{\partial u_x}{\partial y}$$

[98]

**Figure 11** Definition sketch of the velocity gradient produced by shear of a linearly viscous fluid in the $xy$-plane.
Substituting eqn [98] into eqn [96] yields

$$\tau_{yz} = 2\mu\Lambda_{yz}$$ \hspace{1cm} [99]$$

For an isotropic, incompressible fluid eqn [99] can be generalized to three dimensions as

$$\tau_{ij} = 2\mu\Lambda_{ij}$$ \hspace{1cm} [100]$$

The tensor defined by eqn [100] is called the viscous stress tensor; the similarity of the expression between stress and rate of distortion in a linearly viscous material and between stress and strain in a linearly elastic body (e.g., eqns [78] and [87]) is evident. In a manner similar to that for a linearly elastic body, a more general treatment of the relationship between stresses and rates of distortion in a linearly viscous fluid can be derived, including coefficients analogous to Lamé’s constants (e.g., Middleton and Wilcock, 1994).

Let us consider further the stresses acting in a viscous fluid. For a fluid at rest, we find from the definition of the rate-of-distortion tensor that $$\Lambda_{ij} = 0$$. Hence, for a viscous fluid at rest the viscous stress tensor is zero and, therefore, the stress components $$\tau_{xx} = \tau_{yy} = \tau_{zz} = 0$$. However, even in a static fluid, a normal stress equal to the hydrostatic pressure ($$p$$) acts on all fluid particles. Therefore, the total stress acting on a viscous fluid is composed of two parts: a hydrostatic pressure and a viscous stress. Thus, for a viscous fluid, the total stress tensor can be written as

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$ \hspace{1cm} [101]$$

The reason for writing the viscous stress tensor as $$\tau_{ij}$$ to explicitly distinguish it from the notation used for the total stress tensor, $$\sigma_{ij}$$, should now be apparent. The reader may now recognize the similarity between the total stress tensor defined for a viscous fluid and the general total stress tensor previously defined in terms of the mechanical mean pressure (mean normal stress) and the deviatoric stress, $$\sigma_{ij} = -p\delta_{ij} + a_{ij}^m$$.

The thermodynamic hydrostatic pressure $$p$$ and the viscous stress tensor $$\tau_{ij}$$ are, in the general case, unequal to the mechanical mean pressure and the deviatoric stress (Malvern, 1969; Middleton and Wilcock, 1994). For an incompressible fluid, however, the mechanical mean pressure is identical to the hydrostatic pressure, and hence the viscous stress tensor is identical to the deviatoric stress. Even for a monatomic gas, the mechanical mean pressure and hydrostatic pressure are nearly the same and so are the viscous and deviatoric stresses (Middleton and Wilcock, 1994). For an isotropic, incompressible, linearly viscous fluid, we can write eqn [101] as

$$\sigma_{ij} = -p\delta_{ij} + 2\mu\Lambda_{ij}$$ \hspace{1cm} [102]$$

which represents the surface forces acting on a linearly viscous fluid element.

Equation [102] can be used with Cauchy’s first law, the conservation of momentum expression, to derive sets of equations that describe pressure and velocity components of a flowing isothermal fluid. Inserting eqn [102] into eqn [74] (or eqn [100] into eqn [76]), we can write the $$x$$-component for the conservation of momentum of an incompressible linearly viscous fluid as

$$\rho \frac{Du_x}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left[ \frac{2\mu}{\partial x} \frac{\partial u_x}{\partial x} + \frac{\partial}{\partial y} \left( \mu \frac{\partial u_x}{\partial y} \right) \right] + \frac{\partial}{\partial z} \left( \mu \frac{\partial u_x}{\partial z} \right) + \rho g_x$$ \hspace{1cm} [103]$$

With some algebraic manipulation and under the assumption that the viscosity of the fluid is constant, this expression can be recast as

$$\rho \frac{Du_x}{Dt} = -\frac{\partial p}{\partial x} + \mu \left[ \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \right] + \frac{\partial}{\partial x} \left( \mu \frac{\partial u_x}{\partial x} + \frac{\partial u_x}{\partial y} + \frac{\partial u_x}{\partial z} \right) + \rho g_x$$ \hspace{1cm} [104]$$

For an incompressible fluid, conservation of mass dictates that the bracketed third term on the right-hand side of the equation, known as the divergence of the velocity field, is zero. Thus, eqn [104] can be further simplified to

$$\rho \frac{Du_x}{Dt} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u_x + \rho g_x$$ \hspace{1cm} [105]$$

This expression is the $$x$$-direction component of an equation known as the ‘Navier–Stokes equation’ for an incompressible, linearly viscous fluid of constant viscosity. The viscous stress term on the right-hand side of eqn [105] can be written more compactly using the Laplacian operator $$\nabla^2$$ as

$$\rho \frac{Du_x}{Dt} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u_x + \rho g_x$$ \hspace{1cm} [106]$$

This expression can be generalized to three dimensions as

$$\rho \frac{Du_x}{Dt} = -\nabla p + \mu \nabla^2 u + \rho g$$ \hspace{1cm} [107]$$

The Navier–Stokes equation, derived using the conservation of momentum equation, is a control volume representation of Newton’s second law for an incompressible, linearly viscous fluid. Therefore, eqn [107] states that the inertial acceleration (left-hand term) equals the sum of forces (right-hand terms) acting on the system. Because the left-hand term has dimensions of force per unit volume, it is convenient to think of this dynamical equation as a balance of static forces (Tritton, 1988). Thus, the left-hand term can be viewed as an inertial force, and the right-hand terms as pressure, viscous, and body forces, respectively. If we now introduce the concept of the Reynolds number ($$Re$$), which represents the ratio of inertial forces to viscous forces in a fluid, we can highlight some limiting applications of the Navier–Stokes equation. For low-Re flows, viscous forces greatly exceed inertial forces; thus, the left-hand term is negligibly small and eqn [107] can be simplified as

$$\nabla p = \mu \nabla^2 u + \rho g$$ \hspace{1cm} [108]$$

This equation, known as Stokes’ equation, chiefly balances pressure and viscous forces, and describes creeping flow.
been used in geoscience, for example, to obtain estimates of the velocity of a magma as it ascends through the lithosphere (Turcotte and Schubert, 2002). For high-Re flows viscous forces are assumed to be relatively unimportant, and eqn [107] can be simplified to

$$\frac{Du}{Dt} = -\nabla p + \rho g$$  [109]

which is known as Euler’s equation. This equation, which chiefly balances inertial and pressure forces, has applications, for example, in river hydraulics.

### 7.3.8.3 Plasticity – the Coulomb Failure Rule

One of the principal empiricisms in soil mechanics that is used widely in geomorphology relates the mean shear strength acting on a potential failure surface in a soil mass to soil cohesion, normal stress, and the angle of internal friction of the soil. This empiricism, commonly referred to as Coulomb’s law or Coulomb’s failure rule, is generally written as

$$\tau = C + \sigma \tan \phi$$  [110]

where $\tau$ is the mean shearing stress, $C$ the apparent material cohesion (nonfrictional component of the soil strength), $\sigma'$ the effective normal stress (negative in compression) acting on the potential failure surface, and $\phi$ characterizes the friction among soil particles and is called the angle of internal friction of the soil. Apparent soil cohesion depends on electrostatic forces that act between clay particles, on cementation of soil particles owing to secondary mineralization, on surface tension in water films between particles, and on the strength of roots that infiltrate soil surfaces (Selby, 1982; Iverson et al., 1997; Schmidt et al., 2001). The dominant control on soil (and rock) strength, however, is frictional resistance between particles and the interlocking among particles, and the product $\sigma' \tan \phi$ determines the frictional component of shear strength. In general, apparent cohesion of soils is small and not an important contributor to soil strength except in very clay-rich soils, in near-surface soil pervasively penetrated by roots, or in soils where effective stresses are low.

The effect of pore-water pressure on the shear strength of soil is made explicit by substituting the expression for effective stress eqn [18] into eqn [110], which gives

$$\tau = C + (\sigma + p) \tan \phi$$  [111]

This deceptively simple expression is commonly used to assess the factors that govern slope failure. However, this expression is incomplete in that it does not account for the stress and pore-pressure fields that determine the mean shear stress, effective stress, and pore-fluid pressure acting on a potential failure surface (Iverson et al., 1997). The magnitude and spatial distribution of pore-fluid pressure (which is related to the distribution of hydraulic head) and the spatial distribution of solid-grain stress determines the Coulomb failure potential of a soil (Iverson and Reid, 1992; Iverson et al., 1997). Methods for measuring rock and soil strength, along with typical values of cohesion and angles of internal friction, are provided by Selby (1982) and Middleton and Wilcock (1994).

### 7.3.9 Example Application

The concepts discussed in this chapter are widely used in geomorphology, and more broadly in geosciences. The reader is referred to textbooks by Johnson (1970), Means (1976), Middleton and Wilcock (1994), Turcotte and Schubert (2002), and Anderson and Anderson (2010) for several detailed examples. Here, one example relevant to hillslope geomorphology, namely an analysis of slope-failure potential, which utilizes many of the concepts discussed, is presented.

#### 7.3.9.1 Conservation of Momentum and Stress Equilibrium

Iverson and Reid (1992) developed an elastic effective-stress model for gravity-driven groundwater flow and slope-failure potential. To develop the model, they considered an elastic, porous medium that is isotropic and isothermal, assumed that the pore water is isothermal and has uniform density and viscosity, and restricted the model to plane strain and two-dimensional groundwater flow. They began their mathematical formulation by developing equations for stress equilibrium. In terms of coordinates in the $x$- and $y$-directions, they wrote Cauchy’s first law (eqn [74]) for a porous medium with steady fluid flow (stresses are independent of time and there are no inertial accelerations) as

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = -\rho g_x$$

$$\frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{yy}}{\partial x} = -\rho g_y$$  [112]

where $\rho_i$ is the bulk density of the fluid-saturated porous medium and other terms are as previously defined.

#### 7.3.9.2 Effective Stress and Effective Stress Equilibrium

Because the model is for a porous medium that contains pore water, Iverson and Reid (1992) needed to consider effective stress. For this model, they adopted a general definition of effective stress as

$$\sigma'_i = \sigma_i + \beta p d_i$$  [113]

in which $p$ is the pore-water pressure, and $i$ and $j$ the indices that represent $x$ or $y$. We shall return to the coefficient $\beta$ shortly. Substituting eqn [113] into eqn [112], they wrote the stress equilibrium equations in terms of effective stresses as

$$\frac{\partial \sigma'_{xx}}{\partial x} + \frac{\partial \sigma'_{xy}}{\partial y} = -\rho g_x + \beta \frac{\partial p}{\partial x}$$

$$\frac{\partial \sigma'_{yx}}{\partial y} + \frac{\partial \sigma'_{yy}}{\partial x} = -\rho g_y + \beta \frac{\partial p}{\partial y}$$  [114]

By choosing the $y$-coordinate to be positive upward, gravity acts exclusively in the negative $y$-direction, and they defined the hydraulic head in the medium as (see eqn [17])

$$h = \frac{p}{\rho_w g} + y$$  [115]
in which $\rho_w$ is the pore-water density and $g$ the magnitude of gravitational acceleration. Solving for $p$ and substituting into eqn [114] yields effective-stress equilibrium equations with body forces represented in terms of hydraulic head gradients and buoyant unit weight

$$
\frac{\partial \sigma_{xx}^*}{\partial x} + \frac{\partial \sigma_{yy}^*}{\partial y} = \frac{\partial h}{\partial x}
$$

$$
\frac{\partial \sigma_{yy}^*}{\partial x} + \frac{\partial \sigma_{xx}^*}{\partial y} = \left( \rho_i - \rho \rho_w \right) g + \rho \rho_w \frac{\partial h}{\partial y}
$$

where $\left( \rho_i - \rho \rho_w \right) g$ represents the hydrostatic buoyancy effects of the pore fluid, and the head gradient terms represent the seepage force associated with pore-fluid flow.

### 7.3.9.3 Effective Stress and Elastic Strain

Equations [116] contain three unknown stress components. To determine these stress components, it is necessary to identify an appropriate constitutive relationship that relates stresses to strains in the solid porous medium. In this particular problem, Iverson and Reid (1992) adopted constitutive equations for a porous elastic medium developed by Biot (1941), which were recast in terms of conventional elastic moduli by Rice and Cleary (1976). Note, however, that eqn [116] is cast in terms of effective stress and not total stress. Because of this, the constitutive equations need to be cast in terms of effective stress as well. Iverson and Reid (1992) showed that Biot’s constitutive equations for total stress reduced to Hooke’s-law-type equations for effective stress only if the coefficient $\beta$ in eqn [113] is defined as $\beta = 1 - \kappa_b / \kappa_s$, where $\kappa_b$ and $\kappa_s$ are the elastic bulk moduli for the solid porous medium and the individual solid grains, respectively. Using that definition for $\beta$ leads to the following constitutive expressions relating elastic (plane) strains to effective stresses:

$$
e_{xx} = \frac{1}{E} \left[ (1 - \nu) \sigma_{xx}^* - \nu (1 + \nu) \sigma_{yy}^* \right]
$$

$$
e_{yy} = \frac{1}{E} \left[ (1 - \nu) \sigma_{yy}^* - \nu (1 + \nu) \sigma_{xx}^* \right]
$$

$$
e_{xy} = \frac{1}{E} \sigma_{xy}^*
$$

[117]

These expressions, which are identical to the standard constitutive expressions for a linear elastic material except that they are cast in terms of effective stresses, show that effective stress alone is responsible for the elastic deformation of the porous medium.

Slope failure potential is typically assessed using the Coulomb failure rule. Numerous observations of material failure show that the definition of effective stress most useful for describing Coulomb failure is given by eqn [18] (Skempton, 1960; Lambe and Whitman, 1969; Iverson and Reid, 1992). The discrepancy between that definition of effective stress and the one used for defining the constitutive relations for a poroelastic material (eqn [113]) can be eliminated if we assume that the constituent particles forming the porous medium are much less compressible than the porous medium as a whole (i.e., $\kappa_b \ll \kappa_s$). In that case $\beta$ = 1, and eqn [113] approximates eqn [18] (Iverson and Reid, 1992). Such an assumption is reasonable for near-surface Earth materials and allows eqn [117] to be interpreted in terms of Coulomb failure potential.

### 7.3.9.4 Displacement Formulation of Constitutive Relations and Groundwater Flow

The stress equilibrium and constitutive equations for stresses and strains are commonly solved subject to appropriate boundary conditions for stress. However, the problem tackled by Iverson and Reid (1992) dictates that boundary conditions be specified in terms of displacements rather than stresses. They, therefore, recast the governing equations in terms of effective stresses and displacements. This is achieved by substituting the expressions relating strains to displacements (eqns [29], [30], and [39]) into eqn [117] and manipulating the result to obtain

$$
\sigma_{xx}^* = \frac{E}{1 + \nu} \left( \frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_y}{\partial y^2} \right) + \frac{\nu E}{(1 + \nu) (1 - 2\nu)} \left( \frac{\partial^2 U_u}{\partial x^2} + \frac{\partial^2 U_v}{\partial y^2} \right)
$$

$$
\sigma_{yy}^* = \frac{E}{1 + \nu} \left( \frac{\partial^2 U_y}{\partial y^2} + \frac{\partial^2 U_x}{\partial x^2} \right) + \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \left( \frac{\partial^2 U_u}{\partial x^2} + \frac{\partial^2 U_v}{\partial y^2} \right)
$$

[118]

$$
\sigma_{xy}^* = \frac{E}{2 (1 + \nu)} \left( \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_y}{\partial x^2} \right)
$$

Substituting eqn [118] into eqn [116] with $\beta = 1$, followed by further algebraic manipulation, results in two equations for the two components of displacement in terms of the elastic moduli $E$ and $\nu$:

$$
\frac{E}{2 (1 + \nu)} \left( \frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_y}{\partial y^2} \right) + \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \left( \frac{\partial^2 U_u}{\partial x^2} + \frac{\partial^2 U_v}{\partial y^2} \right)
$$

$$
= \rho \rho_w \frac{\partial h}{\partial x} - \rho \rho_w \frac{\partial h}{\partial y}
$$

[119]

Recognizing the coefficients in this set of equations as Lamé’s constants, eqns [119] can be written as

$$
G \left( \frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_y}{\partial y^2} \right) + [\lambda + G] \left( \frac{\partial^2 U_u}{\partial x^2} + \frac{\partial^2 U_v}{\partial y^2} \right) = \rho \rho_w \frac{\partial h}{\partial x}
$$

$$
G \left( \frac{\partial^2 U_y}{\partial y^2} + \frac{\partial^2 U_x}{\partial x^2} \right) + [\lambda + G] \left( \frac{\partial^2 U_u}{\partial x^2} + \frac{\partial^2 U_v}{\partial y^2} \right)
$$

= (\rho_i - \rho) g + \rho \rho_w \frac{\partial h}{\partial y}
$$

[120]
The formulation thus far provides two equations in three unknowns \((U_x, U_y, h)\). Thus, another expression is required to solve the problem. *Iverson and Reid (1992)* closed the problem by using the conventional equation for steady, two-dimensional groundwater flow in a hydraulically isotropic porous medium, as given by

\[
\frac{\partial}{\partial x} \left( K \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left( K \frac{\partial h}{\partial y} \right) = 0 \tag{121}
\]

where \(K\) is the hydraulic conductivity, a function of the permeability of the medium and the viscosity of the pore fluid (e.g., *Freeze and Cherry, 1979*).

Equations [120] and [121] provide a mathematically complete problem that can be solved subject to appropriate boundary conditions in terms of displacements and hydraulic head gradients. *Iverson and Reid (1992)* and *Reid and Iverson (1992)* imposed the following boundary conditions: a periodic surface topography; no forces acting on the topographic surface; a bottom boundary at a large, but finite depth; no flow boundaries along the lateral and basal margins of the domain; and hydraulic head at any point along the topography surface equal to the elevation of the point. Numerical solution of the mathematical problem subject to these boundary conditions showed that gravity-driven groundwater flow produces a spatially variable body force field that influences the effective-stress distribution in hillslopes; that gravitational effects alone produce topographically influenced elastic stresses that can generate near-surface failures even in the absence of groundwater flow; and that near-surface failure potential is increased as a result of groundwater flow, especially near the slope toe. In addition to these broad conclusions, *Iverson and Reid (1992)* discussed insightful subtleties of the mathematical formulation of the problem posed.

### 7.3.10 Concluding Remarks

In this chapter, a brief survey of key concepts of continuum mechanics is provided. The topics discussed covered total stress, pore-fluid pressure, effective stress, strain, conservation, and a few idealized constitutive relationships. The fundamental concepts were derived from considerations of mass, motion, and geometry in an effort to show the physical foundations that lie behind the sometimes complex mathematical formulations. We hope that exposing the underlying physical principles of these concepts and laying out the derivations in some detail provides the reader an entrance into the literature that discusses these topics in greater detail, and that it equips the reader with a suitable foundation to better understand theoretical papers published in geomorphology.

### References


Biographical Sketch

Jon Major is a research hydrologist with the US Geological Survey Cascades Volcano Observatory in Vancouver, Washington. He received his PhD in 1996 from the Department of Geological Sciences at the University of Washington. His research focuses on hydrogeomorphic responses to landscape disturbance, particularly in volcanic river systems. He has worked on groundwater flow in landslides, mechanics of deposition by debris flows, post-eruption sediment transport and streamflow hydrology, hydrogeomorphic response to dam removal, and analyses of debris flow and flood hazards at volcanoes in Washington, Oregon, Alaska, El Salvador, Chile, and the Philippines. He is a fellow of the Geological Society of America (GSA), and has received the GSA EB Burwell Award (Engineering Geology Division research publication award), the GSA Kirk Bryan Award (Quaternary Geology and Geomorphology Division research publication award), and a US Department of Interior Award for Excellence of Service.