STABILITY OF SPHERICALLY CONFINED NANODROPS

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ABSTRACT. We study the geometry of a stable drop of incompressible liquid constrained to lie in a spherical container. The energy functional is comprised of the surface tension, the wetting energy and the line tension. It is shown that the only stable equilibrium drops having the topology of a disc are flat discs and spherical caps.

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The determination of the equilibrium surface having free boundary on a supporting surface is a subject with a long and interesting history which probably originates with Da Vinci’s investigation of the surface of a capillary tube. The basic three phase system consists of a partially filled container, in the present case, a spherical container partially filled by a liquid. Josiah Gibbs observed that points on the free boundary curve which is given by the interface of the liquid’s surface with the container, being in contact with three distinct phases, should carry an energy term of their own; the so called line tension. For isotropic materials, this energy is represented by a multiple of the length of the boundary curve. Since line tension scales linearly while the surface energy scales quadratically, the line tension
becomes increasingly more significant at small scales. In addition, at small scales the gravitational force which scales cubically, is negligible.

In this note, we study the stability of equilibria for a liquid drop contained within a spherical container. Since the drop is liquid, the energy of its free surface $\Sigma$ is proportional to the area and we normalize the constant of proportionality to be one. The interaction of the material of the drop with that of the container contributes a wetting energy which is proportional to the area of the spherical region $\Omega$ in contact with the bulk of the liquid. Points on the free boundary curve $\partial \Sigma$ are in contact with three phases; the drop, the container and the material, assumed to be air, which occupies the region interior to the sphere which is complimentary to the drop. These points contribute an energy term, the so called line tension, which is proportional to the length of $\partial \Sigma$. We thus arrive at the total energy

\begin{equation}
E := \text{Area}[\Sigma] + \omega \text{Area}[\Omega] + \beta \text{Length}[\partial \Sigma],
\end{equation}

where $\omega$ and $\beta$ are coupling constants. The main result presented here is the following:

**Theorem 0.1.** Let $X : (\Sigma, \partial \Sigma) \rightarrow (B^3, S^2)$ be a $C^2$ immersed equilibrium drop where $\Sigma$ is the unit disc in $\mathbb{R}^2$. Then, if the surface is stable, $X(\Sigma)$ is a spherical cap or a flat disc.

To be clear, we do not show here that the smoothness assumption holds a priori, nor do we classify which spherical caps and flat discs are stable. The second of these tasks requires a somewhat lengthy analysis.

Nitsche [8] considered the free boundary problem without wetting or line tension, ($\omega = \beta = 0$), and gave a beautiful complex analytic argument to show that the only equilibrium disc type solutions, stable or otherwise, are spherical caps and flat discs. In [9], Ros and Souam extended this result to drops with wetting energy but no line tension ($\beta = 0$).

For genus zero surfaces contained in the upper half space $x_3 \geq 0$ and having one free boundary component in the plane $x_3 = 0$, it is known that for $\beta \geq 0$, all equilibrium surfaces are spherical caps [6]. When $\beta < 0$ holds, it was shown by B. Widom [12], that the equilibrium spherical caps are never stable. These surfaces are however energy minimizing among rotationally symmetric surfaces. It was pointed out in [11] that this mathematical instability does not preclude the physical existence of drops with negative line tension, since the wave lengths of destabilizing variations may lie below the length scale for which this type of energy model is valid.

In order to obtain the equations characterizing equilibrium, we subject the surface $\Sigma$ to a variation $X_\epsilon = X + \epsilon \delta X + O(\epsilon^2)$. Letting $X$ denote the position vector of the surface, we write $\delta X = T + \psi N$, where $N$ denotes the surface normal and $T$ is tangent to $\Sigma$. The field $\delta X$ must, in addition, be tangent to the sphere $S^2$ along $\partial \Sigma$, i.e.

\begin{equation}
X \cdot \delta X \equiv 0, \text{ on } \partial \Sigma
\end{equation}
and it must satisfy the condition

\[ \int_{\Sigma} \psi \, d\Sigma = 0 \]

which means that the variation infinitesimally preserves the volume of the drop. A standard argument using the Implicit Function Theorem [2] can be applied to show that conditions (2) and (3) are sufficient to produce an actual variation \( X + \epsilon \delta X + O(\epsilon^2) \) which preserves the volume and keeps the drop confined to the sphere.

The first variation of the total energy is

\[ \delta \text{Area}[\Sigma] = -\int_{\Sigma} 2H\psi \, d\Sigma + \oint_{\partial \Sigma} \delta X \cdot n \, ds , \]

where \( n = X' \times N \) is the unit conormal to \( \partial \Sigma \) which it tangent to \( \Sigma \). ("prime" denotes differentiation with respect to arc length along \( \partial \Sigma \)) and \( H \) is the (scalar) mean curvature. The variation of the wetting energy is [9]

\[ \delta \text{Area}[\Omega] = \oint_{\partial \Sigma} \delta X \cdot \tilde{n} \, ds , \]

where \( \tilde{n} = X \times X' \) is the unit conormal to \( \partial \Sigma \) in \( S^2 \). We point out that our orientation of the surface \( \Omega \) is different from that of the surface \( \Sigma \) since \( n \times X' = N \), while \( \tilde{n} \times X' = -X \). Along \( \partial \Sigma \), there holds \( X'' = -X + \tilde{k}_g \tilde{n} \), where \( \tilde{k}_g \) is the geodesic curvature of \( \partial \Sigma \) in \( S^2 \). The variation of the line tension is

\[ \delta \text{Length}[\partial \Sigma] = -\oint_{\partial \Sigma} X'' \cdot \delta X \, ds . \]

By considering variations with compact support satisfying (3), we can conclude that \( H \equiv \text{constant} \) in the interior of \( \Sigma \). Then, collecting the boundary terms, we conclude that

\[ 0 = \oint_{\partial \Sigma} (n + \omega \tilde{n} + \beta(X - \tilde{k}_g \tilde{n})) \cdot \delta X \, ds \]

must hold for all admissible \( \delta X \). This will hold if and only if \( n + (\omega - \beta \tilde{k}_g)\tilde{n} + \beta X \) is parallel to \( X \) along \( \partial \Sigma \), i.e.

\[ X \times (n + (\omega - \beta \tilde{k}_g)\tilde{n}) = 0 \]

must hold. Since \( X \times \tilde{n} = -X' \) and \( X \times n = (X \cdot N)N \times n = (X \cdot N)X' \), we see that (7) is equivalent to

\[ X \cdot N = -\beta \tilde{k}_g + \omega , \]

and the first variation is given by

\[ \delta E = -\int_{\Sigma} 2H\psi \, d\Sigma + \oint_{\partial \Sigma} [X \cdot N + \beta \tilde{k}_g - \omega]X' \times X \cdot \delta X \, ds . \]

Partial examples, i.e. constant mean curvature surfaces having a boundary arc on the sphere where (8) holds, can be produced using Bjorling’s Formula [5]. Let \( C : I \to S^2 \) be curve which is real analytic in its arc length parametrization such that its geodesic curvature satisfies \( | -\beta \tilde{k}_g + \omega | < 1 \) for constants \( \omega, \beta \). We also
use $k_g$ to denote the analytic extension of the geodesic curvature to a neighborhood of $I$ in the complex plane. Then

$$X(z = u + iv) := \text{Re}\left( C(z) - i \int_{z_0}^z (\omega - \beta k_g) C(\zeta) \times C'(\zeta) + \sqrt{1 - (\beta k_g - \omega)^2} C(\zeta) d\zeta \right)$$

defines a minimal surface satisfying (8) on $C[I]$.

The second variation of the functional $E$ was worked out in detail in the paper [10], however we do the calculations for the special case where the supporting surface is a sphere in the appendix. Assuming the surface $\Sigma$ to be in equilibrium, the second variation of energy is given by

$$\delta^2 E = -\int_{\Sigma} \psi L[\psi] d\Sigma + \oint_{\partial \Sigma} \psi B[\psi] ds.$$  

Here $L = \Delta + (4H^2 - 2K)$, where $K$ is the Gaussian curvature of the surface and $\Delta$ denotes the Laplace-Beltrami operator. With $\sin \alpha := X \cdot n$, the boundary operator is has the form

$$B[\psi] = \nabla \psi \cdot n - \left( \frac{1}{\sin \alpha} - \cot \alpha \ dN(n) \cdot n \right) \psi - \frac{\beta}{\sin \alpha} \left( \frac{\psi}{\sin \alpha} ss + \frac{\psi}{\sin \alpha} \right).$$

At points on the boundary where $X \cdot n =: \sin \alpha = 0$, $\psi / \sin \alpha$ must be replaced by $-T \cdot n$, where $T$ is the tangential part of $\delta X$. Defining $\tilde{\psi} := \psi|_{\partial \Sigma} / \sin \alpha$ and integrating by parts, the second variation can now be expressed

$$\delta^2 E = \int_{\Sigma} |\nabla \psi|^2 - |dN|^2 \psi d\Sigma + \oint_{\partial \Sigma} \beta \left( (\tilde{\psi}_s)^2 - \tilde{\psi}^2 \right) - \psi^2 \sec \alpha - \cot(\alpha) II(n, n) ] ds$$

This formula extends (10) to the class of functions

$$\mathcal{K} := \{ \psi \in H^1(\Sigma) | \tilde{\psi} := \psi / \sin \alpha \in H^1(\partial \Sigma) \}.$$

**Definition** An equilibrium surface is stable if $\delta^2 E \geq 0$ holds for all $\psi \in \mathcal{K}$ such that (3) holds.

Diagonalizing the second variation leads to a spectral problem of the form

$$(L + \lambda) \psi = 0, \text{ on } \Sigma, B[\psi] = 0 \text{ on } \partial \Sigma.$$  

The second condition is a type of Wentzell boundary condition, has been widely studied [3], [7].

It will be assumed that $\Sigma$ is an embedded topological disc which is in equilibrium for the functional $E$ such that $\partial \Sigma$ is a piecewise smooth curve. Our aim is to show that if $\Sigma$ is stable, then $\Sigma$ must be axially symmetric. For $c \in (\mathbb{R}^3)^*$, the field $c \times X$ generates an infinitesimal rotation with axis $c$. It is clear that this flow, being a one parameter family of isometries, preserves volumes, areas and arc lengths. Introduce the function $\psi_c := c \times X \cdot N$. It is then clear that $\psi_c$ satisfies $L[\psi_c] = 0$ in $\Sigma$ with $B[\psi_c] = 0$ on $\partial \Sigma$. Also note that (3) holds since rotations preserve volume and (2) is obviously satisfied. We will need the following:

**Lemma 0.2.** If there exists an arc $\gamma$ in the boundary of the disc on which $\psi_c \equiv 0$ holds, then the surface is axially symmetric and it must be a flat disc or a spherical cap.
Proof. Assuming that the hypotheses of the lemma hold, at least one of the following two statements must be true: (i) there is an open arc $\gamma_1 \subset \gamma$ on which $X \cdot n$ is nowhere zero. (ii) there is an open arc $\gamma_1 \subset \gamma$ on which $X \cdot n \equiv 0$ holds.

In case (i), write

$$0 \equiv \psi_c = c \times X \cdot N = c \times X \cdot n \times X' = -(X \cdot n)(c \cdot X'),$$

so $c \cdot X' \equiv 0$ holds on $\gamma_1$. It follows that $X(\gamma_1)$ lies in the intersection of $S^2$ with a plane, so it is a circular arc.

Since $X(\gamma_1)$ is a circular arc, $\bar{k}_g$ is constant and we get from (8)

$$0 = \partial_s(X \cdot N + \beta \bar{k}_g - \omega) = \partial_s(X \cdot N) = X \cdot (-\tau_g n - k_N X') = -\tau_g X \cdot n,$$

where $\tau_g$ denotes the geodesic torsion of $\partial \Sigma$ in $\Sigma$ and $k_N$ denotes the normal curvature. This means that $\tau_g \equiv 0$ holds on $\gamma_1$. We use this to compute the normal derivative of $\psi_c$ on $\gamma_1$:

$$\partial_n \psi_c = c \cdot n \cdot N + c \cdot X \cdot dN(n)$$

$$= -N \cdot n \cdot c - (2H - k_N) c \times X \cdot n - \tau_g c \times X \cdot X'$$

$$= -X' \cdot c - (2H - k_N) X' \cdot c - \tau_g c \times X \cdot X'$$

$$= 0.$$  

Now we have that $\psi_c \equiv 0 \equiv \partial_n \psi_c$ along $\gamma_1$. The Hopf Lemma can then be applied to show that $\psi_c \equiv 0$ holds. Although the Hopf Lemma is usually stated for equations in divergence form $\nabla \cdot A \nabla u = 0$, it can be modified to apply to the operator $L$ in the following way. Let $N_0 := N(p_0)$ for $p_0 \in \partial \Sigma$. Then $L[N \cdot N_0] = 0$ holds and $N \cdot N_0 > 0$ holds near $p_0$. Write $\psi_c = f \cdot N_0$ near $p_0$. Then $L[\psi] = 0$ is equivalent to the divergence form equation $\nabla \cdot (N \cdot N_0)^2 \nabla f = 0$. Also, $\psi_c(p) = 0$ if and only if $f = 0$ and if $f(p) = 0$ for $p \in \partial \Sigma$, then $\partial_n \psi_c(p) = (N \cdot N_0)^2 \partial_n f(p)$. Applying the Hopf Lemma to $f$ gives the result.

If $\psi_c \equiv 0$ holds, then $\Sigma$ is axially symmetric about the vector $c$, since the torque field $X \times c$ is everywhere tangent to the surface, so each of its integral curves, which are coaxial circles perpendicular to $c$, are contained in the surface. Since the surface has constant mean curvature and is assumed to be a topological disc, $\Sigma$ must either be a spherical cap or a flat disc.

If case (ii) holds, we must have $X \cdot N \equiv \pm 1$ on $\gamma_1$, since $X \cdot X' \equiv 0$ holds. Consequently $N' = \pm X'$ implies that $k_N \equiv \pm 1$ and $\tau_g \equiv 0$ holds. We again get from (8) that $\bar{k}_g \equiv constant$ on $X(\gamma_1)$. Since the curvature $\kappa$ of $X(\gamma_1)$ as a space curve satisfies $\kappa^2 = 1 + \bar{k}_g^2$, $\kappa \equiv constant$ holds. We can then easily obtain from $X'' = -X + \bar{k}_g \bar{n}$, that $X'''$ and $X'$ are parallel. Since the torsion of $X(\gamma_1)$ is given $X' \times X'' \cdot X''' / |X' \times X''|$, the torsion is zero and so $X(\gamma_1)$ is a circular arc.

Note that when $X \cdot n \equiv 0$ holds, a calculation similar to that done in (13) shows that $\psi_a := a \times X \cdot N$ vanishes identically along $\gamma_1$ for any non zero vector $a$. We take $a$ to be a non zero vector which is perpendicular to the plane containing $\gamma_1$. The same steps in the calculation for $\partial_n \psi_c$ done above then show that $\partial_n \psi_a \equiv 0$ holds along $\gamma_1$ and by the same reasoning, the surface is axially symmetric and must be a spherical cap or flat disc. q.e.d
Proof of Theorem 0.1 On $\partial \Sigma$, there holds $\psi_c = c \times X \cdot N = -N \times X \cdot c = -(X \cdot n)N \times n \cdot c = -(X \cdot n)X' \cdot c$. We will use this to show that if we assume the surface is not axially symmetric, then there always exists a $c \in (\mathbb{R}^3)^*$ such that the function $\psi_c$ has, at least, four sign changes on $\partial \Sigma$.

If $(r, \theta)$ are the polar coordinates on the disc, then $X' = X_\theta/|X_\theta|$ and so
\begin{equation}
\psi_c|_{\partial \Sigma} = -(X \cdot n/|X_\theta|)|X_\theta| \cdot c.
\end{equation}

For any $c$ holds. Let
\begin{equation}
\oint_{\partial \Sigma} X_\theta \cdot c \, d\theta = \oint_{\partial \Sigma} (X \cdot c)_\theta \, d\theta = 0
\end{equation}

Thus, there exists $c \in \mathbb{R}^3$, $c \neq 0$ with $0 = c \cdot A = c \cdot B$. It then follows that for this $c$ the function $X_\theta \cdot c$ can be represented as a Fourier series of the form
\begin{equation}
X_\theta \cdot c = \sum_{j \geq 2} (a_j \cos(j\theta) + b_j \sin(j\theta)).
\end{equation}

This function can be interpreted as the boundary values of the real part of the complex analytic function
\begin{equation}
F(z) := \sum_{j \geq 2} (a_j - ib_j)z^j.
\end{equation}

Note that $F(z) = z^2\eta(z)$ for a function $\eta(z)$ which is complex analytic in the disc. The variation of $\arg(F(z))$ over the boundary of the disc is therefore at least $4\pi$. If $\Re(F(z))$ vanishes identically on an open arc in $\partial \Sigma$, then the same is true for $\psi_c$ and, by Lemma (0.2), the surface is axially symmetric so we can assume that this does not happen. Because the variation of $\arg(F(z))$ is at least $4\pi$, $\Re(F(z))$ must have at least four sign changes on $\partial \Sigma$. Specifically, there are at least four points $p_i$ which are contained in arcs on which $\Re(F(z))$ is positive on one side of $p_i$ and negative on the other side. The same is true for the function $\psi_c$. This follows from equation (15) and the fact that $X \cdot n \geq 0$ holds on $\partial \Sigma$ since the function $\|X\|^2$ clearly assumes its maximum on $\partial \Sigma$. Also recall that $X \cdot n$ cannot vanish on any arc in $\partial \Sigma$ by Lemma 0.2 and (15).

At each point $p_i$, at least one arc of the nodal set of $\psi_c$ must enter $\Sigma$. These curves must divide the disc into at least three nodal domains of the function $\psi_c$. We will now show that the existence of more than two nodal domains of $\psi_c$ implies instability. Suppose $\Omega_1, \Omega_2, \ldots, \Omega_N$ are nodal domains, $N \geq 3$. We can assume $\Omega_1$ and $\Omega_2$ are adjacent to each other. Let $U = \Omega_1 \cup \Omega_2$, $V = \Sigma \setminus U$.

Define $\mathcal{F}$ to be the set of all functions $f$ on $\bar{U}$ satisfying the following conditions: (i) $f$ is piecewise $C^1$ on $U$, (ii) $f/X \cdot n$ is piecewise $C^1$ on $\partial \Sigma$ and (iii) $f \equiv 0$ on $\partial U \setminus \partial \Sigma$. Note that $\psi_c|_U \in \mathcal{F}$. Also, since $(X \cdot N)^2 + (X \cdot n)^2 \equiv 1$ on $\partial \Sigma$, $\mathcal{F}$ contains
all functions vanishing identically on V which are of the form \(v(1 - (X \cdot N)^2)\) near \(\partial U \cap \partial \Sigma\), where \(v\) is smooth function. In particular, this includes \(C^\infty_c(U)\). Define

\[
\mu_1 = \inf_{\mathcal{F}} \left( \int_U |\nabla f|^2 - (4H^2 - 2K)f^2 \, d\Sigma - \int_{\partial \Sigma \cap \partial U} \left( \frac{1}{\sin \alpha} - \cot \alpha \, dN(n) \cdot n \right) f^2 \, ds \right.
\]

\[
\left. + \beta \int_{\partial \Sigma \cap \partial U} \left( \hat{f}^2 - \tilde{f}^2 \right) \, ds \right) / \left( \int_U f^2 \, d\Sigma \right).
\]

By using the function \(\psi \in \mathcal{F}\), we get that \(\mu_1 \leq 0\) holds.

Suppose \(\mu_1 = 0\). Let \(\psi^*\) be the function which is identically \(\psi_e\) in \(\Omega_1\) and is identically zero off \(\Omega_1\). Then \(\psi^*\) realizes the infimum \(\mu_1 = 0\) and hence we have, for all \(\zeta \in C^\infty_c(U)\),

\[
0 = \partial_\epsilon \left( \int_U |\nabla (\psi^* + \epsilon \zeta)|^2 - (4H^2 - 2K)(\psi^* + \epsilon \zeta)^2 \, d\Sigma \right.
\]

\[
\left. - \int_{\partial \Sigma \cap \partial U} \left( \frac{1}{\sin \alpha} - \cot \alpha \, dN(n) \cdot n \right)(\psi^* + \epsilon \zeta)^2 \, ds \right.
\]

\[
\left. + \beta \int_{\partial \Sigma \cap \partial U} (\hat{\psi}^* + \epsilon \hat{\zeta})^2 - (\hat{\psi}^* + \epsilon \hat{\zeta})^2 \, ds \right)_{\epsilon = 0}
\]

\[
= 2 \int_U \nabla \psi^* \cdot \nabla \zeta - (4H^2 - 2K)\psi^* \zeta \, d\Sigma.
\]

In other words, \(\psi^*\) is a weak solution of \(L = 0\) in \(U\). Elliptic regularity then implies that \(\psi^*\) is a classical solution. However \(\psi^* \equiv 0\) on \(\Omega_2\) which contradicts a well known continuation property [1].

We can therefore conclude that \(\mu_1\) is negative and so there is then an \(f \in \mathcal{F}\) for which the ratio in (16) is negative. Extend \(f\) to be zero in \(V\). Let \(\psi_2\) be the function which is identically equal to \(\psi_e\) in \(V\) and is identically zero in \(U\). There is a nontrivial superposition \(\phi := c_1 f + c_2 \psi_2 \in \mathcal{K}\) which has zero mean value. We seek a variation field \(\delta X = \phi N + T\), with \(T\) tangent to \(\Sigma\) such that on \(\partial \Sigma\) there holds \(0 \equiv \delta X \cdot X = \phi N \cdot X + (T \cdot n)(n \cdot X)\). In other words, we want \(T \cdot n = -\phi/(n \cdot X)\). This last expression is well-defined and piecewise differentiable by the definition of \(\mathcal{F}\). Let \(w\) be the solution of the biharmonic Dirichlet problem \(\Delta^2 w = 0\) in \(\Sigma\) having boundary values \(w \equiv 0\) and \(\partial_n w = -\phi/(n \cdot X)\). Then setting \(\delta X := \nabla w + \phi N\), we get \(\delta X \cdot X \equiv 0\) on \(\partial \Sigma\). Using a standard method ([2]), this variation field can be shown to arise from a genuine one parameter family of surfaces contained in the ball which enclose the same volume as \(\Sigma\) and for this variation, the second variation of energy would be negative. Thus, for the surface to be stable \(\psi_e \equiv 0\) must hold, in which case the surface is axially symmetric.

**q.e.d.**

1. **Appendix: The Second Variation Formula**

We consider a one parameter family of embeddings

\[
I \times (\Sigma, \partial \Sigma) \rightarrow (B^3, S^2)
\]

\[(t, p) \mapsto X_t(p),\]
where $I = (-\epsilon, \epsilon)$ and $X \equiv X_0$. The variation fields $\delta_{t}^{(n)}(X_t)_{t=0}$ will be denoted by $\delta^N X$. As before, we express $\delta X$ in terms of normal and tangential components as $\psi N + T$.

It is assumed that the volume between the surface $X_t(\Sigma)$ and the 2-sphere is constant. In addition to (Refvp), this implies the second order condition

$$0 = \int_{\Sigma} \delta^2 X \cdot N \, d\Sigma + \int_{\Sigma} \delta X \cdot \delta(N d\Sigma).$$

Using this and the fact that $H \equiv \text{constant}$, we compute the variation of the surface integral in (4).

$$-\delta \int_{\Sigma} 2H \delta X \cdot N \, d\Sigma = -\int_{\Sigma} 2(\delta H) \delta X \cdot N \, d\Sigma = -\int_{\Sigma} \psi L[\psi] \, d\Sigma.$$

We have used here that $\delta H = (1/2)L[\psi] + \nabla H \cdot T = L[\psi]$ since the mean curvature is constant when $t = 0$. We can then deduce from (9) that, since the term in square brackets vanishes when $t = 0$, the second variation of energy is given by

$$\delta^2 E = -\int_{\Sigma} \psi L[\psi] \, d\Sigma + \int_{\partial \Sigma} \delta \left[ X \cdot N + \beta \bar{k}_g - \omega \delta X \cdot \bar{n} \right] \cdot \delta X \, ds$$

$$= -\int_{\Sigma} \psi L[\psi] \, d\Sigma - \int_{\partial \Sigma} \delta \left[ X \cdot N + \beta \bar{k}_g \right] \bar{n} \cdot \delta X \, ds$$

It is clear that the $T$ does not contribute to the variation of the surface integral (4) and it is also clear from (5) and (6) that the component $\delta X \cdot X'$ does not contribute to the variations of the energy $E$. We can therefore assume that

$$\delta X \cdot X' = 0$$

holds on $\partial \Sigma$ which simplifies the calculations.

Along $\partial \Sigma$ we have two right handed bases $X, X', \bar{n}$ and $n, X', N$. We can therefore write $X = \cos \alpha \, N + \sin \alpha \, n$, $\bar{n} = \sin \alpha \, N - \cos \alpha \, n$. From $\delta X \cdot X = 0$, we have $\psi \cos \alpha + T \cdot n \sin \alpha = 0$ and we can define

$$\hat{\psi} = \begin{cases} \psi \csc \alpha, & \text{if } \sin \alpha \neq 0, \\ -T \cdot n & \text{if } \sin \alpha = 0. \end{cases}$$

Using (RefdX), the first order change in the frame $X, X', \bar{n}$ is given by

$$\delta X = \hat{\psi} \bar{n}, \delta X' = \hat{\psi}' \bar{n}, \delta \bar{n} = -\hat{\psi} X - \hat{\psi}' X'.$$

The first variation of $\bar{k}_g$ is then given by

$$\delta \bar{k}_g = -\delta \left( \frac{\bar{n}' \cdot X'}{X' \cdot X} \right)$$

$$= -(\delta \bar{n}') \cdot X' - \bar{n}' \cdot \delta X' + 2 \delta X' \cdot X'$$

$$= \hat{\psi}'' + \hat{\psi}.$$
Here we have used that when \( t = 0 \), \( s \) is the arc length parameter along \( \partial \Sigma \). We also need
\[
\delta (X \cdot N) = \delta X \cdot N + X \cdot \delta N
\]
\[
= \psi + X \cdot (\nabla \psi + dN(T))
\]
\[
= \psi - \psi_n \sin \alpha + (X \cdot n)((T \cdot n)dN(n) \cdot n)
\]
\[
= \psi - \psi_n \sin \alpha - \psi \cos \alpha \ dN(n) \cdot n
\]
Combining this with \( \delta X \cdot \hat{n} = \psi / \sin \alpha \), gives
\[
\delta^2 E = - \int_\Sigma \psi L[\psi] \ d\Sigma + \int_{\partial \Sigma} \psi \left( \psi_n - \psi \left( \frac{1}{\sin \alpha} - \cot \alpha \ dN(n) \cdot n \right) \right) \ ds - \beta \int_{\partial \Sigma} \hat{\psi} \left( \hat{\psi}'' + \hat{\psi} \right) \ ds.
\]
Note that at points where \( \sin \alpha = 0 \), we can make sense of the integrand by replacing \( \psi / \sin \alpha \) with \( \hat{\psi} \) defined by (19). Also, in the case \( \beta = 0 \), this formula agrees with that found in [9].

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