

## DETC2000/MECH-14069

### DIMENSIONAL SYNTHESIS OF BENNETT LINKAGES

**Alba Perez**

Robotics and Automation Laboratory  
 Department of Mechanical and Aerospace Engineering  
 University of California  
 Irvine, California 92697  
 Email: maperez@uci.edu

**J. M. McCarthy\***

Robotics and Automation Laboratory  
 Department of Mechanical and Aerospace Engineering  
 University of California  
 Irvine, California 92697  
 Email: jmmccart@uci.edu

#### ABSTRACT

*This paper presents a synthesis procedure for a spatial 4R linkage, known as Bennett's linkage. It is known that the two solutions of the RR chain synthesis equations form a Bennett linkage. While analytical solutions to these equations have been developed previously, this paper uses the cylindroid that is known to exist for a Bennett linkage to simplify the solution process. It is interesting that geometric constraint associated with the spatial 4R chain simplifies the solution of the RR chain design equations. An example design is presented.*

#### 1 Introduction

In this paper we present a procedure to design spatial 4R chains known as Bennett linkages, using the synthesis equations for the spatial RR chain. Veldkamp (Veldkamp, 1967) solved the design equations of a spatial RR chain for three instantaneous positions, and showed that the two solutions form a Bennett linkage. Tsai and Roth (Tsai and Roth, 1973) solved the 10 quadratic equations for three position synthesis using the screw triangle formulation and showed that it always has two solutions which form a Bennett linkage.

We use Huang's result (Huang, 1996) that the finite displacement screws of a Bennett linkage form a cylindroid to determine a coordinate frame in which the Tsai and Roth's design equations simplify. The result is three linear equations and one cubic polynomial in four design parameters.

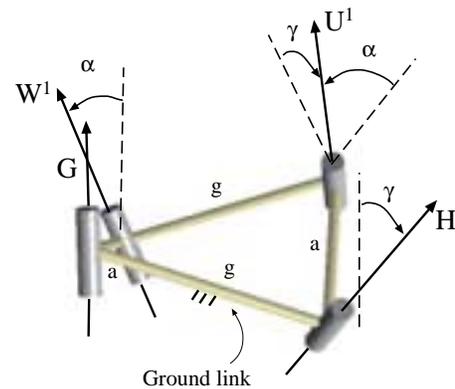


Figure 1. A Bennett linkage

#### 2 The Bennett Linkage

The Bennett linkage is formed by connecting the end-links of two spatial RR chains to form a coupler, Figure 1. In order to move, the twist angles and link lengths of the opposites sides of this linkage must be equal. Let the angle of twist and length of the two cranks be  $\alpha$ ,  $a$  and  $\gamma$ ,  $g$  for the ground link and coupler. Then Bennett showed that the condition

$$\frac{\sin \alpha}{a} = \frac{\sin \gamma}{g}, \quad (1)$$

ensures that the linkage moves with one degree of freedom.

To design the Bennett linkage we use the design equations for an RR chain. The RR chain consists of a fixed revolute axis

\*Address all correspondence to this author.

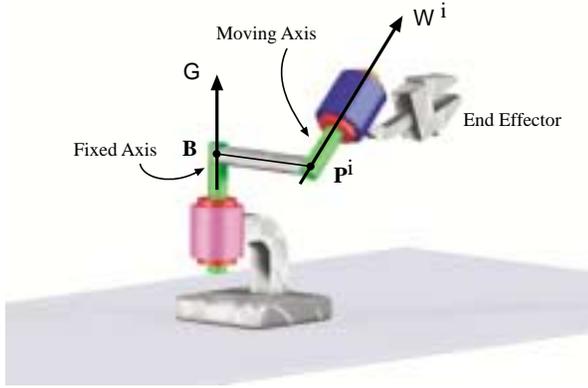


Figure 2. A spatial RR robot.

$\mathbf{G} = (\mathbf{G}, \mathbf{B} \times \mathbf{G})^T = (\mathbf{G}, \mathbf{R})^T$  connected to a moving revolute axis  $\mathbf{W}^i = (\mathbf{W}^i, \mathbf{P}^i \times \mathbf{W}^i)^T = (\mathbf{W}^i, \mathbf{V}^i)^T$  by a rigid link. The vector  $\mathbf{G}$  denotes the direction of the fixed axis and  $\mathbf{B}$  is a point on this line. The vector  $\mathbf{W}^i$  is the direction of the moving axis in the  $i$ th position and  $\mathbf{P}^i$  a point on this line. We choose  $\mathbf{B}$  and  $\mathbf{P}^i$  to be the intersection points of these lines with the common normal  $\mathbf{N}$  to the two axes, Figure 2.

In order to design the RR chain we need to determine the directions  $\mathbf{G}$  and  $\mathbf{W}^i$  of the fixed and moving axes and the coordinates of the points  $\mathbf{B}$  and  $\mathbf{P}^i$ . This is a total of 10 design parameters. The solution of the design equations yields two RR chains which form the Bennett linkage.

### 3 Geometry of the RR Chain

The set of relative screw displacements reachable by the RR chain is achieved by a spatial rotation by  $\phi$  about the  $\mathbf{W}^i$  axis followed by a rotation by  $\theta$  about the  $\mathbf{G}$  axis. Let  $\hat{G}(\frac{\theta}{2}) = \cos(\frac{\theta}{2}) + \sin(\frac{\theta}{2})\mathbf{G}$  and  $\hat{W}^i(\frac{\phi}{2}) = \cos(\frac{\phi}{2}) + \sin(\frac{\phi}{2})\mathbf{W}^i$  be the dual quaternions associated with these rotations (Bottema and Roth, 1990), (McCarthy, 1990). The dual quaternion  $\hat{S}$  which represents the displacement of the end-link is obtained by the dual quaternion product  $\hat{S} = \hat{G}\hat{W}^i$ , which yields the dual scalar

$$\cos(\frac{\hat{\Psi}}{2}) = \cos\frac{\theta}{2}\cos\frac{\phi}{2} - \sin\frac{\theta}{2}\sin\frac{\phi}{2}\mathbf{G} \cdot \mathbf{W}^i, \quad (2)$$

and the dual vector

$$\sin(\frac{\hat{\Psi}}{2})\mathbf{S} = \sin\frac{\theta}{2}\cos\frac{\phi}{2}\mathbf{G} + \sin\frac{\phi}{2}\cos\frac{\theta}{2}\mathbf{W}^i + \sin\frac{\theta}{2}\sin\frac{\phi}{2}\mathbf{G} \times \mathbf{W}^i. \quad (3)$$

The dual angle  $\hat{\Psi}$  defines the rotation and translation of the end-link along the screw axis  $\mathbf{S}$ . This can be written in a useful form

by dividing the screw (3) by the dual scalar (2) to obtain

$$\tan(\frac{\hat{\Psi}}{2})\mathbf{S} = \frac{\tan\frac{\theta}{2}\mathbf{G} + \tan\frac{\phi}{2}\mathbf{W}^i + \tan\frac{\theta}{2}\tan\frac{\phi}{2}\mathbf{G} \times \mathbf{W}^i}{1 - \tan\frac{\theta}{2}\tan\frac{\phi}{2}\mathbf{G} \cdot \mathbf{W}^i}. \quad (4)$$

By varying the joint angles  $\theta$  and  $\phi$  in (3) we obtain a two dimensional set of screw axes which define the set of positions reachable by the RR chain.

If, on the other hand, the values of  $\theta$  and  $\phi$  are coupled by the Bennett linkage, then Hunt (Hunt, 1978) shows that

$$\tan\frac{\phi}{2} = K \tan\frac{\theta}{2}, \quad (5)$$

where  $K$  is a constant obtained from the dimensions  $\alpha$  and  $\gamma$ . Substitute (5) into (4) and divide by  $\tan\frac{\theta}{2}$  to obtain

$$\tan(\frac{\hat{\Psi}}{2})\mathbf{S} = \frac{\mathbf{G} + K\mathbf{W}^i + K \tan\frac{\theta}{2}\mathbf{G} \times \mathbf{W}^i}{\cot\frac{\theta}{2} - K \tan\frac{\theta}{2}\mathbf{G} \cdot \mathbf{W}^i}. \quad (6)$$

Remarkably, this equation generates a set of screw axes that form a cylindroid, Figure 3. It is useful to note that the screw axes defined by (3) with the angular relation (5) form the same cylindroid; the division by the dual scalar does not affect the geometry of this system.

A cylindroid is a ruled surface which appears as generated by the real linear combination of two screw axes. Equation 5 defines the relation between the coupler and input angle for a Bennett linkage, hence the cylindroid is the locus of relative screw displacements for a Bennett linkage.

On the following sections we will relate the cylindroid generated by the Bennett linkage with the one that contains the relative screws  $\mathbf{S}_{12}$  and  $\mathbf{S}_{13}$ .

### 4 The Cylindroid

A cylindroid is a ruled surface formed by the axes of a real linear combination of two screws. Assume that we have designed an RR chain that passes through three spatial positions  $M_1$ ,  $M_2$  and  $M_3$ . Then the relative screw axes  $\mathbf{S}_{12}$  and  $\mathbf{S}_{13}$  must lie on the cylindroid defined by (6). In fact these two screws must generate this cylindroid. This is the key to our formulation of the RR design problem.

To describe the geometry of the cylindroid, let us consider the screws obtained from the design positions. For this calculation it is convenient to work with the vector part of the quaternion product (3) and define the screws

$$\mathbf{V}_a = \sin\frac{\hat{\Psi}_{12}}{2}\mathbf{S}_{12}, \quad \mathbf{V}_b = \sin\frac{\hat{\Psi}_{13}}{2}\mathbf{S}_{13}. \quad (7)$$

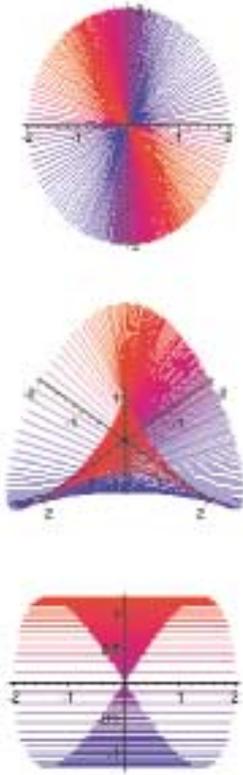


Figure 3. The cylindroid as viewed first from the top along the central axis, then from an angle and finally, from the side.

The dual number  $\sin \frac{\hat{\Psi}}{2} = (\sin \frac{\Psi}{2}, \frac{t_i}{2} \cos \frac{\Psi}{2})$  contains the magnitude and pitch of the screw,

$$\sin \frac{\hat{\Psi}}{2} = \sin \frac{\Psi}{2}(1, P), \quad (8)$$

where the pitch  $P$  is given by the expression

$$P_a = \frac{t_{12}}{2 \tan \frac{\Psi_{12}}{2}}, \quad P_b = \frac{t_{13}}{2 \tan \frac{\Psi_{13}}{2}}. \quad (9)$$

The two screws  $V_a$  and  $V_b$  are independent and generate a cylindroid. To simplify the following calculations denote the screw axis  $S_{12}$  as  $I$  and the common normal line to  $S_{12}$  and  $S_{13}$  as  $K$ . Then we have  $J = K \times I$  to form a frame  $I, J, K$ , and

$$S_{12} = I, \quad S_{13} = \cos \hat{\delta} I + \sin \hat{\delta} J. \quad (10)$$

where  $\hat{\delta} = (\delta, d)$  is the dual angle between  $S_{12}$  and  $S_{13}$ .

The real linear combination of  $V_a$  and  $V_b$  yields

$$F = a(1, P_a)I + b(1, P_b)(\cos \hat{\delta} I + \sin \hat{\delta} J), \quad (11)$$

where we have absorbed  $\sin \frac{\Psi_{12}}{2}$  and  $\sin \frac{\Psi_{13}}{2}$  into the constants  $a$  and  $b$ , respectively. The axes of the screws  $F$  form the cylindroid.

## 5 The Principal Axes

The cylindroid consists of pairs of lines that intersect a central axis that is the common normal to all of the lines. This surface has a set of *principal axes* consisting of the common normal and the only pair of lines that form an angle of 90 degrees. This occurs at the midpoint of the central axis. The following formulation combines the results of (Hunt, 1978) and (Parkin, 1997) to define the principal axes.

We locate the axis of the general screw  $F$  of the cylindroid using the dual angle  $\hat{\zeta} = (\zeta, z)$ , that is

$$F = F(1, P)(\cos \hat{\zeta} I + \sin \hat{\zeta} J), \quad (12)$$

where  $F$  is the magnitude of this screw and  $P$  is its pitch. Expand (11) to obtain

$$F = (a + b \cos \delta, aP_a + b(P_b \cos \delta - d \sin \delta))I + (b \sin \delta, b(P_b \sin \delta + d \cos \delta))J \quad (13)$$

and equate to (12). We separate the real and dual parts of this result,

$$F \begin{Bmatrix} \cos \zeta \\ \sin \zeta \end{Bmatrix} = \begin{bmatrix} 1 & \cos \delta \\ 0 & \sin \delta \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} \quad (14)$$

and

$$F \begin{bmatrix} -\sin \zeta & \cos \zeta \\ \cos \zeta & \sin \zeta \end{bmatrix} \begin{Bmatrix} z \\ P \end{Bmatrix} = \begin{bmatrix} P_a & P_b \cos \delta - d \sin \delta \\ 0 & P_b \sin \delta + d \cos \delta \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix}. \quad (15)$$

Now we eliminate the scalars  $a$  and  $b$  using both equations to obtain

$$F \begin{Bmatrix} cz(\zeta) \\ P(\zeta) \end{Bmatrix} = \begin{bmatrix} -\sin \zeta & \cos \zeta \\ \cos \zeta & \sin \zeta \end{bmatrix} \begin{bmatrix} P_a & (P_b - P_a) \cot \delta - d \\ 0 & P_b + d \cot \delta \end{bmatrix} \begin{Bmatrix} \cos \zeta \\ \sin \zeta \end{Bmatrix} \quad (16)$$

This equation gives the pitch  $P$  and distance  $z$  of the screw for any value of the angle  $\zeta$ . We use this expression to locate

the principal axes of the cylindroid, as the screws with maximum and minimum value of pitch. Computing the derivative of the above expression we obtain

$$\frac{d}{d\zeta} \begin{Bmatrix} z \\ P \end{Bmatrix} = \begin{bmatrix} -(P_b - P_a) \cot \delta + d & (P_b - P_a) + d \cot \delta \\ (P_b - P_a) + d \cot \delta & (P_b - P_a) \cot \delta - d \end{bmatrix} \begin{Bmatrix} \sin 2\zeta \\ \cos 2\zeta \end{Bmatrix} \quad (17)$$

Set the second of these equations to zero to determine the angles  $\zeta = \sigma$  for extreme values of pitch  $P$ , given by

$$\tan 2\sigma = \frac{-(P_b - P_a) \cot \delta + d}{(P_b - P_a) + d \cot \delta}. \quad (18)$$

This yields two angles separated by  $\pi/2$ . They define the principal axes  $X$  and  $Y$  of the cylindroid.

The offset  $z(\sigma)$  at which the principal axes are located is given by

$$z_0 = \frac{d - (P_b - P_a) \cos \delta}{2 \sin \delta} \quad (19)$$

The principal axes can be expressed, using the dual angle  $\hat{\sigma} = (\sigma, z_0)$  as

$$X = \cos \hat{\sigma} I + \sin \hat{\sigma} J, \quad Y = -\sin \hat{\sigma} I + \cos \hat{\sigma} J. \quad (20)$$

The principal axes capture the symmetry of the cylindroid. By expressing the synthesis problem in this coordinate frame we obtain simpler design equations.

## 6 The Design Equations

The design equations for the RR chain are obtained from the geometrical constraints imposed by the link connecting the moving and fixed axes. Let  $\rho$  and  $r$  denote the angle and distance between these axes. Clearly they must remain constant during the movement. Furthermore, the axes can not slide, so that the normal line to both axes remains the same. These constraints can be found in Suh and Radcliffe (Suh and Radcliffe, 1978), (McCarthy, 2000). Tsai and Roth (Tsai and Roth, 1973) use a similar formulation based on the equivalent screw triangle.

Let the three positions be defined by the  $4 \times 4$  homogeneous transforms  $[T_i] = [A_i, \mathbf{d}_i]$ ,  $i = 1, 2, 3$  where  $[A_i]$  is a  $3 \times 3$  rotation matrix and  $\mathbf{d}_i$  is a  $3 \times 1$  translation vector. Construct the relative displacements  $[T_{1i}] = [T_i][T_1^{-1}]$ ,  $i = 2, 3$ . Associated with each of these matrices is a  $6 \times 6$  coordinate transformation for screws,

$$[\hat{T}_{1i}] = \begin{bmatrix} A_{1i} & 0 \\ D_{1i}A_{1i} & A_{1i} \end{bmatrix}, \quad (21)$$

where  $D_{1i}$  is the  $3 \times 3$  skew-symmetric matrix obtained such that  $[D_{1i}]\mathbf{y} = \mathbf{d}_{1i} \times \mathbf{y}$ .

Dual vector algebra now allows to define one set of constraint equations for the RR chain such that

$$\mathbf{G} \cdot [\hat{T}_{1i} - I]\mathbf{W}^1 = 0, \quad i = 2, 3, \quad (22)$$

which is the same equation that is used for CC chains. Separating this dual equation into real and dual components we obtain the direction constraint equations

$$\mathbf{G} \cdot [A_{1i} - I]\mathbf{W}^1 = 0, \quad i = 2, 3, \quad (23)$$

and the moment constraint equations

$$\mathbf{G} \cdot [A_{1i} - I]\mathbf{V}^1 + \mathbf{W}^1 \cdot [A_{1i} - I]^T \mathbf{R} + \mathbf{G} \cdot [D_{1i}A_{1i}]\mathbf{W}^1 = 0, \quad i = 2, 3. \quad (24)$$

These equations ensure that the dual angle  $\hat{\rho} = (\rho, r)$  is constant in the three positions.

In order to ensure that the common normal passes through the same points  $\mathbf{B}$  and  $\mathbf{P}^i = [T_{1i}]\mathbf{P}^1$ , we require  $\mathbf{P}^i - \mathbf{B}$  to be perpendicular to  $\mathbf{G}$  and  $[T_{1i}]^{-1}\mathbf{B} - \mathbf{P}^1$  to be perpendicular to  $\mathbf{W}^1$ , that is

$$\mathbf{G} \cdot ([T_{1i}]\mathbf{P}^1 - \mathbf{B}) = 0, \quad \mathbf{W}^1 \cdot (\mathbf{P}^1 - [T_{1i}]^{-1}\mathbf{B}) = 0, \quad i = 1, 2, 3. \quad (25)$$

The 10 equations (23), (24), (25) are solved to determine  $\mathbf{G}$  and  $\mathbf{W}^1$ .

## 7 The Equivalent Screw Triangle

The equivalent screw triangle formulation of Tsai and Roth allows us to write some of the constraint equations in a different way. For each relative transformation we can construct the screw axis  $S_{1i}$  and a rotation angle  $\psi_{1i}$  and a translation  $t_{1i}$ . The equivalent screw triangle defines the relationship between the screw axis  $S_{1i}$  and the fixed and moving axes  $\mathbf{G}$  and  $\mathbf{W}^1$ , Figure 4. Let  $\mathbf{C}_{1i}$  be a reference point on the screw axis  $S_{1i}$ .

The direction constraint in (23) can be reformulated as

$$\tan \frac{\psi_{1i}}{2} = \frac{\mathbf{G} \cdot (\mathbf{S}_{1i} \times \mathbf{W}^1)}{(\mathbf{S}_{1i} \times \mathbf{G}) \cdot (\mathbf{S}_{1i} \times \mathbf{W}^1)} \quad (26)$$

using the equation of the screw triangle.

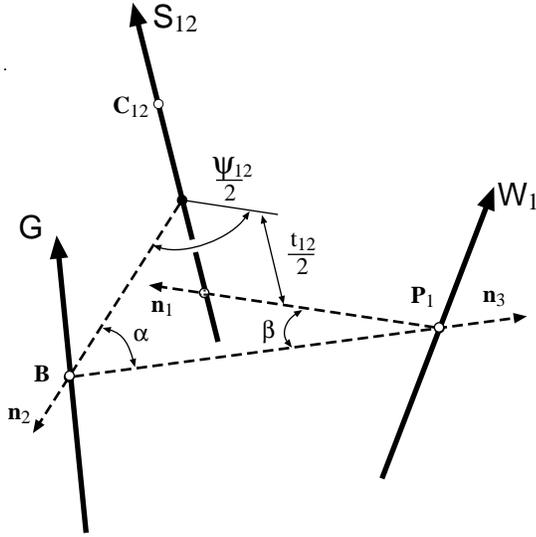


Figure 4. The equivalent screw triangle for the RR chain

The properties of the screw triangle yield an alternative set of equations for the moment constraints. Notice that the geometry of the dyad triangle requires the common normal lines to  $G$  and  $W^i$  to be separated by a distance  $t_{1i}/2$  along  $S_{1i}$ . Thus the component of  $\mathbf{B} - \mathbf{P}^1$  in the direction  $S_{1i}$  is given by

$$(\mathbf{B} - \mathbf{P}^1) \cdot \mathbf{S}_{1i} - \frac{t_{1i}}{2} = 0, \quad i = 2, 3. \quad (27)$$

It is possible to show that these equations are equivalent to (24).

To complete the set of design equations, we transform the equations (25) to obtain

$$\begin{aligned} \mathbf{G} \cdot (\mathbf{P}^1 - \mathbf{B}) &= 0, \\ \mathbf{W}^1 \cdot (\mathbf{P}^1 - \mathbf{B}) &= 0, \end{aligned} \quad (28)$$

and

$$\begin{aligned} \mathbf{G} \cdot [I - \mathbf{S}_{1i} \mathbf{S}_{1i}^T] (\mathbf{B} - \mathbf{C}_{1i}) &= 0, \\ \mathbf{W}^1 \cdot [I - \mathbf{S}_{1i} \mathbf{S}_{1i}^T] (\mathbf{P}^1 - \mathbf{C}_{1i}) &= 0, \quad i = 2, 3. \end{aligned} \quad (29)$$

This is a set of ten quadratic equations in ten unknowns, the coordinates of the line  $G$  and the line  $W^1$  in its first position.

## 8 Bennett Linkage Coordinates

Given three specified positions, we can determine the two relative screws of the displacements, with screw axes  $S_{12}$  and  $S_{13}$ . The principal axes of the cylindroid generated by these two screws provides an efficient form for the design equations.

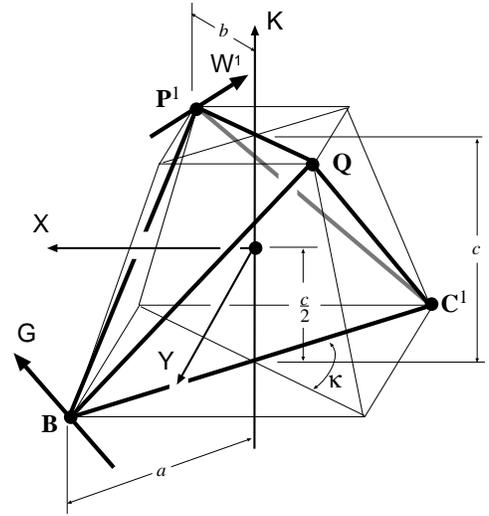


Figure 5. The tetrahedron defines the Bennett linkage.

Yu (Yu, 1981) introduced a coordinate frame aimed to simplify the expression of the Bennett linkage. The joints of the Bennett linkage can be determined using a tetrahedron: each joint passes through one of the four vertices and its direction is perpendicular to the adjacent sides. The construction is shown in Figure 5.

Let  $\mathbf{B}$ ,  $\mathbf{P}^1$ ,  $\mathbf{Q}$ ,  $\mathbf{C}^1$  be the vertices of the tetrahedron. The edges are given by the difference of the vertices. The tetrahedron is oriented so that the  $K$  axis of the cylindroid forms the common normal to the lines defined by  $\mathbf{B} - \mathbf{C}^1$  and  $\mathbf{P}^1 - \mathbf{Q}$  (Huang, 1996). Let  $2a = |\mathbf{B} - \mathbf{C}^1|$  and  $2b = |\mathbf{P}^1 - \mathbf{Q}|$ , and let  $c$  and  $\kappa$  be the distance and angle between both edges along  $K$ . The principal axes  $X$  and  $Y$  are located at a half distance and bisecting the angle  $\kappa$ .

We can completely describe the Bennett linkage in the principal axes frame using the four parameters of the tetrahedron  $a$ ,  $b$ ,  $c$ ,  $\kappa$ . The coordinates of the vertices are given by

$$\begin{aligned} \mathbf{B} &= \begin{Bmatrix} a \cos \frac{\kappa}{2} \\ a \sin \frac{\kappa}{2} \\ -\frac{c}{2} \end{Bmatrix}, & \mathbf{P}^1 &= \begin{Bmatrix} b \cos \frac{\kappa}{2} \\ -b \sin \frac{\kappa}{2} \\ \frac{c}{2} \end{Bmatrix}, \\ \mathbf{Q} &= \begin{Bmatrix} -b \cos \frac{\kappa}{2} \\ b \sin \frac{\kappa}{2} \\ \frac{c}{2} \end{Bmatrix}, & \mathbf{C}^1 &= \begin{Bmatrix} -a \cos \frac{\kappa}{2} \\ -a \sin \frac{\kappa}{2} \\ -\frac{c}{2} \end{Bmatrix}. \end{aligned} \quad (30)$$

To find the direction of the joint axes  $\mathbf{G}$  and  $\mathbf{W}^1$  we compute the cross product of the edges

$$\mathbf{G} = K_g (\mathbf{Q} - \mathbf{B}) \times (\mathbf{P}^1 - \mathbf{B}) = K_g \begin{Bmatrix} 2bc \sin \frac{\kappa}{2} \\ 2bc \cos \frac{\kappa}{2} \\ 4abc \cos \frac{\kappa}{2} \sin \frac{\kappa}{2} \end{Bmatrix} \quad (31)$$

and

$$\mathbf{W}^1 = K_w(\mathbf{B} - \mathbf{P}^1) \times (\mathbf{C}^1 - \mathbf{P}^1) = K_w \begin{Bmatrix} -2ac \sin \frac{\kappa}{2} \\ 2ac \cos \frac{\kappa}{2} \\ 4ab \cos \frac{\kappa}{2} \sin \frac{\kappa}{2} \end{Bmatrix}, \quad (32)$$

where the constants  $K_g$  and  $K_w$  normalize the vectors. The expressions of the screws  $\mathbf{G} = (\mathbf{G}, \mathbf{B} \times \mathbf{G})^T$  and  $\mathbf{W}^1 = (\mathbf{W}^1, \mathbf{P}^1 \times \mathbf{W}^1)^T$  in the principal axis coordinates are

$$\mathbf{G} = K_g \begin{Bmatrix} 2bc \sin \frac{\kappa}{2} \\ 2bc \cos \frac{\kappa}{2} \\ 4ab \cos \frac{\kappa}{2} \sin \frac{\kappa}{2} \end{Bmatrix} + \varepsilon K_g \begin{Bmatrix} b \cos \frac{\kappa}{2} (4a^2 \sin^2 \frac{\kappa}{2} + c^2) \\ -b \sin \frac{\kappa}{2} (4a^2 \cos^2 \frac{\kappa}{2} + c^2) \\ 2abc (\cos^2 \frac{\kappa}{2} - \sin^2 \frac{\kappa}{2}) \end{Bmatrix} \quad (33)$$

and

$$\mathbf{W}^1 = K_w \begin{Bmatrix} -2ac \sin \frac{\kappa}{2} \\ 2ac \cos \frac{\kappa}{2} \\ 4ab \cos \frac{\kappa}{2} \sin \frac{\kappa}{2} \end{Bmatrix} + \varepsilon K_w \begin{Bmatrix} -a \cos \frac{\kappa}{2} (4b^2 \sin^2 \frac{\kappa}{2} + c^2) \\ -a \sin \frac{\kappa}{2} (4b^2 \cos^2 \frac{\kappa}{2} + c^2) \\ 2abc (\cos^2 \frac{\kappa}{2} - \sin^2 \frac{\kappa}{2}) \end{Bmatrix} \quad (34)$$

Similarly, the coordinates of the second RR dyad,  $\mathbf{H}$  and  $\mathbf{U}^1$ , are given by the expressions

$$\begin{aligned} \mathbf{H} &= K_h(\mathbf{C}^1 - \mathbf{Q}) \times (\mathbf{B} - \mathbf{Q}) + \varepsilon K_h \mathbf{Q} \times ((\mathbf{C}^1 - \mathbf{Q}) \times (\mathbf{B} - \mathbf{Q})) \\ \mathbf{U}^1 &= K_u(\mathbf{P}^1 - \mathbf{C}^1) \times (\mathbf{Q} - \mathbf{C}^1) + \varepsilon K_u \mathbf{C}^1 \times ((\mathbf{P}^1 - \mathbf{C}^1) \times (\mathbf{Q} - \mathbf{C}^1)). \end{aligned} \quad (35)$$

Using the Bennett linkage coordinates to define  $\mathbf{G}$  and  $\mathbf{W}^1$ , we reduce the number of design parameters from ten to four. Furthermore, the six equations (28) and (29) are identically satisfied. The result is four equations (26) and (27) in the unknowns  $a$ ,  $b$ ,  $c$  and  $\kappa$ .

## 9 Solving the Design Equations

We now transform the task positions  $[T_i]$  to the principal axis frame and determine the relative screw axes

$$\begin{aligned} S_{12} &= \sin \frac{\hat{\Psi}_{12}}{2} (\cos \hat{\delta}_1 X + \sin \hat{\delta}_1 Y) \\ S_{13} &= \sin \frac{\hat{\Psi}_{13}}{2} (\cos \hat{\delta}_2 X + \sin \hat{\delta}_2 Y), \end{aligned} \quad (36)$$

where  $\hat{\Psi}_{1i} = (\psi_{1i}, t_{1i})$  are the angle and translation of the screw displacements. The dual angles  $\hat{\delta}_i$  are defined relative to the principal axes frame.

Substitute (36) into (27) to obtain two linear equations in  $a$  and  $b$ . They can be solved to obtain

$$a = \frac{K_s}{2 \sin \frac{\kappa}{2}} + \frac{K_d}{2 \cos \frac{\kappa}{2}}, \quad b = \frac{K_s}{2 \sin \frac{\kappa}{2}} - \frac{K_d}{2 \cos \frac{\kappa}{2}}. \quad (37)$$

The constants  $K_s$  and  $K_d$  combine information from the relative screws and are listed in Table 1.

Next we substitute (37) into the set of equations (26) and introduce the usual definition of  $y = \tan \frac{\kappa}{2}$  to eliminate the sine and cosine functions of  $\kappa$ . The result is two rational equations in  $c$  and  $y$ ,

$$\frac{A_{1i}c + 2K_d^2 \tan \frac{\Psi_{1i}}{2} (K_s^2/K_d^2 - y^2)}{((\cos 2\delta_i - 1)y^2 + \cos 2\delta_i + 1)c^2 + 2K_d^2(K_s^2/K_d^2 - y^2)} = 0, \quad (38)$$

where

$$A_{1i} = ((-1 + \cos 2\delta_i)y^2 + \cos 2\delta_i + 1) \tan \frac{\Psi_{1i}}{2} c - 2t_{1i}y, \quad i = 2, 3. \quad (39)$$

We do not clear fractions and solve because the numerator and denominator share two roots associated with  $c = 0$ , that is

$$c = 0, y = \pm \frac{K_s}{K_d}. \quad (40)$$

Note that when  $c = 0$  the linkage is planar.

To eliminate these roots we set the numerators of (38) equal to zero and form the matrix equation

$$\begin{bmatrix} A_{12} & 2K_d^2 \tan \frac{\Psi_{12}}{2} \\ A_{13} & 2K_d^2 \tan \frac{\Psi_{13}}{2} \end{bmatrix} \begin{Bmatrix} c \\ (K_s^2/K_d^2 - y^2) \end{Bmatrix} = 0. \quad (41)$$

In order for this set of equation to have roots other than those given by (40), the determinant of the coefficient matrix must be zero. This yields an equation that is linear in  $c$ . Solve this to obtain

$$c = (K_{13} - K_{12}) \sin \kappa \quad (42)$$

where the constants  $K_{12}$  and  $K_{13}$  are shown in Table 1.

In order to determine  $\kappa$ , we substitute the expressions for  $a$ ,  $b$  and  $c$  into one of the one of the direction equations (26).

| Constant | Expression   |
|----------|--|
| $K_s$    | $\frac{t_{12} \cos \delta_2 - t_{13} \cos \delta_1}{2 \sin(\delta_1 - \delta_2)}$                      |
| $K_d$    | $\frac{t_{13} \sin \delta_1 - t_{12} \sin \delta_2}{2 \sin(\delta_1 - \delta_2)}$                      |
| $K_{12}$ | $\frac{t_{12}/2}{\tan \frac{\psi_{12}}{2}} \left( \frac{1}{\sin^2 \delta_1 - \sin^2 \delta_2} \right)$ |
| $K_{13}$ | $\frac{t_{13}/2}{\tan \frac{\psi_{13}}{2}} \left( \frac{1}{\sin^2 \delta_1 - \sin^2 \delta_2} \right)$ |

Table 1. Constants computed from the specified positions.

Eliminating the trivial solutions with  $c = 0$ , we obtain a cubic polynomial in  $y^2$ ,

$$P: C_3 y^6 + C_2 y^4 + C_1 y^2 + C_0 = 0 \quad (43)$$

with the coefficients given by

$$\begin{aligned} C_3 &= -K_d^2, \\ C_2 &= 4K_s^2 - 8K_d^2 \\ &\quad + (K_{12} - K_{13})(K_{13} \sin^2 \delta_1 - K_{12} \sin^2 \delta_2), \\ C_1 &= 8K_s^2 - 4K_d^2 \\ &\quad - (K_{12} - K_{13})(K_{13} \cos^2 \delta_1 - K_{12} \cos^2 \delta_2), \\ C_0 &= K_s^2. \end{aligned} \quad (44)$$

Substitute  $z = y^2$ , and solve the cubic polynomial to determine its three roots.

This cubic polynomial has only one real positive root for  $z$ . To see this we note that the value of the polynomial at  $z = 0$  is positive, that is  $P(0) = K_s^2$ . For large positive values of  $z$ , the polynomial becomes negative because  $P(\infty) = -K_d^2$ , hence we have at least one positive root. For large negative values of  $z$ , the polynomial remains positive. Now for  $z = -1$ , the value of the polynomial is  $-3K_d^2 - 3K_s^2 - (K_{12} - K_{13})^2$  which is negative for all values of the coefficients. We conclude that there must be two negative roots.

The square root of the positive root gives the two solutions for  $\kappa$ . The result is two sets of solutions  $(a, b, c, \kappa)$  and  $(-b, -a, -c, -\kappa)$ .

The design procedure yields two unique algebraic solutions for the three position synthesis of the RR chain. Due to the symmetry of the principal axes, the second solution given by the values  $(-b, -a, -c, -\kappa)$  corresponds to the coordinates of the RR chain associated with the other two axes of the Bennett linkage, as defined in Equation (35):  $G(a, b, c, \kappa) = -H(-b, -a, -c, -\kappa)$

|                | x   | y   | z   | $\theta$ | $\phi$ | $\psi$ |
|----------------|-----|-----|-----|----------|--------|--------|
| M <sub>1</sub> | 0.0 | 0.0 | 0.0 | 0°       | 0°     | 0°     |
| M <sub>2</sub> | 0.0 | 0.0 | 2.4 | 0°       | 5°     | 40°    |
| M <sub>3</sub> | 3.6 | 2.0 | 0.2 | 19°      | -28°   | 67°    |

Table 2. Three specified positions

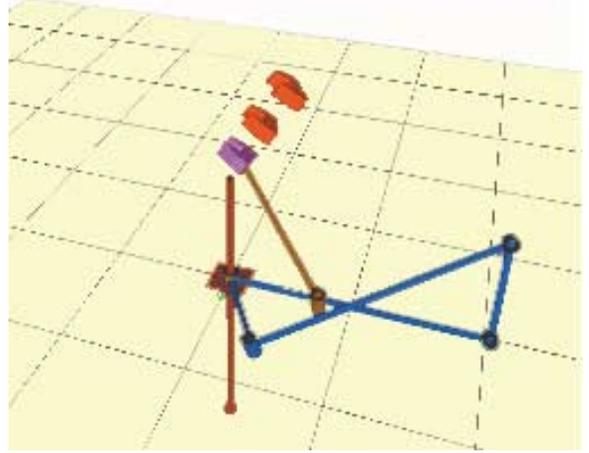


Figure 6. The Bennett linkage

and  $W^1(a, b, c, \kappa) = -U^1(-b, -a, -c, -\kappa)$ . Hence this procedure yields two RR chains, which combine to form a Bennett linkage.

## 10 Example

In Table 2 we present three specified positions, given by the distance  $(x, y, z)$  to the origin and the roll, pitch and yaw orientation angles  $(\theta, \phi, \psi)$  for the moving frame.

We obtain the fixed joint lines G and H and moving joint lines  $W^1$  and  $U^1$ . Table 3 presents their coordinates in the original frame. Figure 6 shows the obtained Bennett linkage.

## 11 Conclusions

This paper presents a new formulation of the solution for the three position synthesis of a Bennett linkage. The procedure combines the results of Tsai and Roth (1973) for the spatial RR chain with the geometric properties of the cylindroid studied by Huang (1996). Using the properties of the cylindroid, we were able to use the principal axis frame of the cylindroid and a new set of design parameters to simplify the original design problem from ten quadratic equations in ten unknowns to four equations in four unknowns. We find algebraic expressions for the

| Axis  | Line coordinates   |
|-------|--|
| $G$   | $(-0.23, 0.49, -0.84)^T + \epsilon(-0.76, 2.75, 1.83)^T$ |
| $W^1$ | $(-0.52, 0.44, -0.73)^T + \epsilon(-2.52, 2.33, 3.22)^T$ |
| $H$   | $(0.57, 0.43, 0.70)^T + \epsilon(2.25, 2.21, -3.21)^T$   |
| $U^1$ | $(0.29, 0.52, 0.80)^T + \epsilon(0.45, 2.54, -1.83)^T$   |

Table 3. Design parameters

unknowns and were able to prove that there two unique solutions that combine to form a Bennett linkage.

## REFERENCES

- Bottema, O., and Roth, B., *Theoretical Kinematics*, North-Holland, 1979, (reprinted by Dover Publications, 1990).
- Huang, C., "The Cylindroid Associated with Finite Motions of the Bennett Mechanism," *Proceedings of the ASME Design Engineering Technical Conferences*, Irvine, CA, 1996.
- Hunt, K.H., *Kinematic Geometry of Mechanisms*. Clarendon Press, 1978.
- McCarthy, J. M., *Introduction to Theoretical Kinematics*. The MIT Press, 1990.
- McCarthy, J. M., *Geometric Design of Linkages*. Springer-Verlag, 2000.
- Parkin, I.A., "Finding the Principal Axes of Screw Systems." *Proceedings of the ASME Design Engineering Technical Conferences*, Sacramento, CA, 1997.
- Suh, C.H. and Radcliffe, C.W., *Kinematics and mechanisms design*. John Wiley & Sons, 1978.
- Tsai, L.W. and Roth, B., "A Note on the Design of Revolute-Revolute Cranks". *Mechanisms and Machine Theory*, 1973, Vol.8, pp 23-31.
- Tsai, L.W., *Design of open-loop chains for rigid body guidance*. Stanford University Ph.D.dissertation, 1972.
- Veldkamp, G.R., "Canonical Systems and Instantaneous Invariants in Spatial Kinematics." *Journal of Mechanisms*, 1967, Vol.3., pp. 329-388.
- Yu, H.C., "The Bennett Linkage, its Associated Tetrahedron and the Hyperboloid of its Axes". *Mechanism and Machine Theory*, Vol. 16, pp. 105-114, 1981.