Dimensional Synthesis of Bennett Linkages

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- The workspace of the RR chain
- The cylindroid
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Literature review

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• Analysis


• Linear Systems of Screws

The Bennett Linkage - Spatial closed 4R chain

- Formed by connecting the end links of two spatial RR chains \{G, W\} and \{H, U\} to form a coupler link.

- Mobility: For a general 4R closed spatial chain:

\[
M = 6(n - 1) - \sum_{k=1}^{m} p_k c_k = 6.3 - 4.5 = -2
\]

However, the Bennett linkage can move with one degree of freedom.

- Special geometry:
  - Link length and twist angle \((a, \alpha)\) and \((g, \gamma)\) must be the same for opposite sides.
  - Ratio of the sine of the twist angle over the link length:

\[
\frac{\sin \alpha}{a} = \frac{\sin \gamma}{g}
\]

- The joints of the Bennett linkage form the vertices of a tetrahedron
**The Design Theory - The Workspace of the RR Chain**

- **Synthesis Theory** - Find the kinematic chain that reaches exactly a number of specified positions

  ![Diagram of an RR chain with labels for fixed axis, moving axis, and end effector]

- The specified positions **must lie on the workspace of the chain**.

- The kinematics equation for the RR chain defines its workspace.
The Kinematics Equation for the RR Chain

- The **kinematics equation in matrix form**: The set of displacements \([D(\theta, \phi)]\) of the RR chain.

\[
[D] = [G][Z(\theta, 0)][X(\alpha, a)][Z(\phi, 0)][H]
\]

- If we choose a reference configuration \([D_1]\), we can write the workspace of the **relative displacements** \([D_{1i}] = [D_i][D_1]^{-1}\).

\[
[D_{1i}] = [T(\theta_i, G)][T(\phi_i, W)]
\]

where

\[
[T(\Delta \theta, G)] = [G][Z(\theta, 0)][Z(\theta_0, 0)]^{-1}[G]^{-1},
\]

\[
[T(\Delta \phi, W)] = ([G][Z(\theta_0, 0)][X(\rho, r)] [Z(\phi_0, 0)] [Z(\phi, 0)]^{-1} ([G][Z(\theta_0, 0)][X(\rho, r)])^{-1}
\]
The Kinematics Equation for the RR Chain
Dual quaternion formulation

- We can also formulate the workspace using **dual quaternions** to express the relative displacements.

The dual quaternion form of the workspace is given by:

\[
\hat{D}_{1i} = \hat{G}(\Delta \theta) \hat{W}(\Delta \phi)
\]

\[
\cos(\frac{\hat{\psi}_{1i}}{2}) = \cos \frac{\Delta \theta}{2} \cos \frac{\Delta \phi}{2} - \sin \frac{\Delta \theta}{2} \sin \frac{\Delta \phi}{2} G \cdot W,
\]

\[
\sin(\frac{\hat{\psi}_{1i}}{2})S_{1i} = \sin \frac{\Delta \theta}{2} \cos \frac{\Delta \phi}{2} G + \sin \frac{\Delta \phi}{2} \cos \frac{\Delta \theta}{2} W + \sin \frac{\Delta \theta}{2} \sin \frac{\Delta \phi}{2} G \times W
\]

where \( S_{1i} \) is the screw axis of the relative displacement and \( \hat{\psi}_{1i} = (\phi_{1i}, d_{1i}) \) is the associated rotation about and slide along this axis for each displacement in the workspace.

- Every pair of values \( \Delta \theta \) and \( \Delta \phi \) defines a screw axis \( S_{1i} \) that represents a relative displacement from position 1 to position \( i \).
The Workspace of the Bennett Linkage

- **Restriction to a Bennett linkage**: The angles $\theta$ and $\phi$ are not independent. There exist the input/coupler angular relation:

\[
\tan \frac{\phi}{2} = -\frac{\sin \frac{\gamma + \alpha}{2}}{\sin \frac{\gamma - \alpha}{2}} \tan \frac{\theta}{2} = K \tan \frac{\theta}{2}
\]

- **The workspace of the Bennett linkage**: The set of screw axes obtained applying the input/coupler relation to the workspace of the RR chain,

\[
\tan \left( \frac{\psi_{1i}}{2} \right) S_{1i} = \frac{G + K'' W^1 + K' \tan \frac{\theta}{2} G \times W^1}{\cot \frac{\theta}{2} - K' \tan \frac{\theta}{2} G \cdot W^1}.
\]

generates a **cylindroid**
The cylindroid

- Simply-Ruled surface that has a nodal line cutting all generators at right angles.

\[ z(x^2 + y^2) + (P_X - P_Y)xy = 0 \]

- It appears as generated by the real linear combination of two screws.
- The cylindroid has a set of principal axes located in the midpoint of the nodal line.

- The principal axes can be located from any pair of generators.

\[ \tan 2\sigma = \frac{-(P_b - P_a) \cot \delta + d}{(P_b - P_a) + d \cot \delta} \]

\[ z_0 = \frac{1}{2} (d - (P_b - P_a) \frac{\cos \delta}{\sin \delta}) \]
Yu, 1981: The Bennett linkage can be determined using a tetrahedron defined by four parameters \((a, b, c, \kappa)\).

The principal axes are located in the middle of the tetrahedron.

The joint axes \(G\) and \(W_1\) are given by the cross product of the edges. This ensures that the chosen points \(B, P_1\) are on the common normal.

\[
G = K_g (Q - B) \times (P^1 - B) + \epsilon K_g B \times ((Q - B) \times (P^1 - B))
\]

\[
W_1 = K_w (B - P^1) \times (C^1 - P^1) + \epsilon K_w P^1 \times ((B - P^1) \times (C^1 - P^1))
\]

We obtain:

\[
G = K_g \begin{bmatrix} 2bc \sin \frac{\kappa}{2} \\ 2bc \cos \frac{\kappa}{2} \\ 4ab \cos \frac{\kappa}{2} \sin \frac{\kappa}{2} \end{bmatrix} + \epsilon K_g \begin{bmatrix} b \cos \frac{\kappa}{2} (4a^2 \sin^2 \frac{\kappa}{2} + c^2) \\ -b \sin \frac{\kappa}{2} (4a^2 \cos^2 \frac{\kappa}{2} + c^2) \\ 2abc (\cos^2 \frac{\kappa}{2} - \sin^2 \frac{\kappa}{2}) \end{bmatrix}
\]

and

\[
W_1 = K_w \begin{bmatrix} -2ac \sin \frac{\kappa}{2} \\ 2ac \cos \frac{\kappa}{2} \\ 4ab \cos \frac{\kappa}{2} \sin \frac{\kappa}{2} \end{bmatrix} + \epsilon K_w \begin{bmatrix} -a \cos \frac{\kappa}{2} (4b^2 \sin^2 \frac{\kappa}{2} + c^2) \\ -a \sin \frac{\kappa}{2} (4b^2 \cos^2 \frac{\kappa}{2} + c^2) \\ 2abc (\cos^2 \frac{\kappa}{2} - \sin^2 \frac{\kappa}{2}) \end{bmatrix}
\]

Using the principal axes and the tetrahedron formulation, we can write the coordinates of the joint axes of the Bennett linkage with only four parameters.
The design equations for an RR dyad

- **The constant dual angle constraint**: $\hat{\alpha} = (\alpha, a)$, the angle and distance between the fixed and moving axes, must remain constant during the movement.

  $$G \cdot [\hat{T}_{1i} - I]W^1 = 0, \ i = 2, 3,$$

Using the equivalent screw triangle formulation and separating real and dual part,

1. **The direction equations**

   $$\tan \frac{\psi_{1i}}{2} = \frac{G \cdot (S_{1i} \times W^1)}{(S_{1i} \times G) \cdot (S_{1i} \times W^1)}, \ i = 2, 3.$$

2. **The distance equations**

   $$(B - P^1) \cdot S_{1i} - \frac{t_{1i}}{2} = 0, \ i = 2, 3.$$

- **The normal constraints**: The normal line to $G$ and $W$, $P^i - B$, remains the same.

  $$G \cdot ([T_{1i}]P^1 - B) = 0,$$

  $$W^1 \cdot (P^1 - [T_{1i}]^{-1}B) = 0, \ i = 1, 2, 3.$$

Total equations: $2(n - 1) + 2n$

Total parameters: 10

Number of positions needed for a finite number of solutions: $n = 3$.

*The standard algebraic formulation of the synthesis problem consists on solving ten equations in ten parameters.*
Solving the design equations in the principal axes frame

- The six common normal constraints are automatically satisfied.
- We solve system of four equations in four parameters $a$, $b$, $c$, $\kappa$.

\[
\tan \frac{\psi_{12}}{2} = \frac{G \cdot (S_{12} \times W^1)}{(S_{12} \times G) \cdot (S_{12} \times W^1)} \tag{1}
\]

\[
\tan \frac{\psi_{13}}{2} = \frac{G \cdot (S_{13} \times W^1)}{(S_{13} \times G) \cdot (S_{13} \times W^1)} \tag{2}
\]

\[
(B - P^1) \cdot S_{12} - \frac{t_{12}}{2} = 0 \tag{3}
\]

\[
(B - P^1) \cdot S_{13} - \frac{t_{13}}{2} = 0 \tag{4}
\]

Solution for $a$ and $b$ : the distance equations (3) and (4) are linear in $a$, $b$.

\[
\frac{t_{12}}{2} + (a - b) \cos \delta_1 \cos \frac{\kappa}{2} + (a + b) \sin \delta_1 \sin \frac{\kappa}{2} = 0
\]

\[
\frac{t_{13}}{2} + (a - b) \cos \delta_2 \cos \frac{\kappa}{2} + (a + b) \sin \delta_2 \sin \frac{\kappa}{2} = 0
\]

Defining the constraints:

\[
K_s = \frac{t_{12} \cos \delta_2 - t_{13} \cos \delta_1}{2 \sin(\delta_1 - \delta_2)}
\]

\[
K_d = \frac{t_{13} \sin \delta_1 - t_{12} \sin \delta_2}{2 \sin(\delta_1 - \delta_2)}
\]

We obtain:

\[
a = \frac{K_s}{2 \sin \frac{\kappa}{2}} + \frac{K_d}{2 \cos \frac{\kappa}{2}}
\]

\[
b = \frac{K_s}{2 \sin \frac{\kappa}{2}} - \frac{K_d}{2 \cos \frac{\kappa}{2}}.
\]
Solution for $c$:

- Substitute the values of $a$ and $b$ in the direction equations (1) and (2) and make the algebraic substitution $y = \tan \frac{\kappa}{2}$.

\[
\tan \frac{\varphi_{12}}{2} (\frac{K^2_{2d}}{K_d} - y^2) + c^2 \tan \frac{\varphi_{12}}{2K_d} (y^2 (\cos 2\delta_1 - 1) + \cos 2\delta_1 + 1) - 2 \frac{y}{K_d} (\cos \delta_1 + \frac{K_s \sin \delta_1}{K_d}) \left( \frac{K^2_{2d}}{K_d} - y^2 \right) + \frac{c^2}{2K^2_d} (y^2 (\cos 2\delta_1 - 1) + \cos 2\delta_1 + 1) = 0,
\]

\[
\tan \frac{\varphi_{13}}{2} (\frac{K^2_{2d}}{K_d} - y^2) + c^2 \tan \frac{\varphi_{13}}{2K_d} (y^2 (\cos 2\delta_2 - 1) + \cos 2\delta_2 + 1) - 2 \frac{y}{K_d} (\cos \delta_2 + \frac{K_s \sin \delta_2}{K_d}) \left( \frac{K^2_{2d}}{K_d} - y^2 \right) + \frac{c^2}{2K^2_d} (y^2 (\cos 2\delta_2 - 1) + \cos 2\delta_2 + 1) = 0,
\]

- The numerator and denominator share the roots associated with $c = 0$, which are not a solution of the spatial problem. Eliminate them from the numerator forcing the linear system to have more solutions than the trivial.

\[
\begin{bmatrix}
\tan \frac{\varphi_{12}}{2} & \tan \frac{\varphi_{13}}{2}
\end{bmatrix}
\begin{bmatrix}
\frac{\tan \frac{\varphi_{12}}{2}}{2K^2_d} (y^2 c (\cos 2\delta_1 - 1) + \cos 2\delta_1 + 1) - 2 \frac{y}{K_d} (\cos \delta_1 + \frac{K_s \sin \delta_1}{K_d}) \\
\frac{\tan \frac{\varphi_{13}}{2}}{2K^2_d} (y^2 c (\cos 2\delta_2 - 1) + \cos 2\delta_2 + 1) - 2 \frac{y}{K_d} (\cos \delta_2 + \frac{K_s \sin \delta_2}{K_d})
\end{bmatrix}
\left\{ \frac{(K^2_{2d}/c - y^2)}{c} \right\} = 0.
\]

- Making the determinant of the matrix equal to zero we obtain a linear equation in $c$.
- Define the constants

\[
K_{12} = \frac{t_{12}/2}{\tan \frac{\varphi_{12}}{2}} \left( \frac{1}{\sin^2 \delta_1 - \sin^2 \delta_2} \right)
\]
\[
K_{13} = \frac{t_{13}/2}{\tan \frac{\varphi_{13}}{2}} \left( \frac{1}{\sin^2 \delta_1 - \sin^2 \delta_2} \right)
\]

- We obtain the expression for $c$:

\[
c = (K_{13} - K_{12}) \sin \kappa
\]
Solution for $\kappa$:

- Substitute the expressions for $a$, $b$, $c$ in one of the direction equations, (1) or (2). We obtain a cubic polynomial in $y^2$.

$$P : \quad C_3 y^6 + C_2 y^4 + C_1 y^2 + C_0 = 0$$

- The coefficients are:

  $$C_3 = -K_d^2,$$
  $$C_2 = K_s^2 - 2K_d^2 + 4(K_{12} - K_{13})(K_{13} \sin^2 \delta_1 - K_{12} \sin^2 \delta_2),$$
  $$C_1 = 2K_s^2 - K_d^2 - 4(K_{12} - K_{13})(K_{13} \cos^2 \delta_1 - K_{12} \cos^2 \delta_2),$$
  $$C_0 = K_s^2.$$  

- Solve the cubic polynomial for $z = y^2$. This polynomial has one and only one real positive root $z_0$:

  $$P(0) = K_s^2$$
  $$P(\infty) = -K_d^2$$
  $$P(-1) = -4(K_{12} - K_{13})^2$$

- The square root of the positive root gives the two solutions for $\kappa$.

  $$\tan \frac{\kappa}{2} = \pm \sqrt{z_0}$$
The Solutions

- The two sets of solutions \((a, b, c, +\kappa)\) and \((-b, -a, -c, -\kappa)\) correspond to both dyads of the Bennett mechanism:

<table>
<thead>
<tr>
<th>Solution 1</th>
<th>Solution 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(G(a, b, c, \kappa))</td>
<td>(H(-b, -a, -c, -\kappa))</td>
</tr>
<tr>
<td>(W(a, b, c, \kappa))</td>
<td>(U(-b, -a, -c, -\kappa))</td>
</tr>
<tr>
<td>(H(a, b, c, \kappa))</td>
<td>(G(-b, -a, -c, -\kappa))</td>
</tr>
<tr>
<td>(U(a, b, c, \kappa))</td>
<td>(W(-b, -a, -c, -\kappa))</td>
</tr>
</tbody>
</table>

- The synthesis procedure yields the two RR dyads that form a Bennett linkage.
Example

Tsai and Roth positions (Tsai and Roth, 1973)

- The specified positions:

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>θ</th>
<th>φ</th>
<th>ψ</th>
</tr>
</thead>
<tbody>
<tr>
<td>M₁</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0°</td>
<td>0°</td>
<td>0°</td>
</tr>
<tr>
<td>M₂</td>
<td>0.0</td>
<td>0.0</td>
<td>0.8</td>
<td>0°</td>
<td>0°</td>
<td>40°</td>
</tr>
<tr>
<td>M₃</td>
<td>1.11</td>
<td>0.66</td>
<td>0.05</td>
<td>18.8°</td>
<td>-28.0°</td>
<td>67.2°</td>
</tr>
</tbody>
</table>

- The joint axes in the initial frame:

<table>
<thead>
<tr>
<th>Axis</th>
<th>Line coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>G</td>
<td>(0.36, 0.45, 0.81), (0.26, 1.05, -0.70)</td>
</tr>
<tr>
<td>W¹</td>
<td>(0.60, 0.36, 0.72), (0.87, 0.83, -1.14)</td>
</tr>
<tr>
<td>H</td>
<td>(0.60, -0.36, 0.72), (0.87, -0.83, -1.14)</td>
</tr>
<tr>
<td>U¹</td>
<td>(0.36, -0.45, 0.81), (0.26, -1.05, -0.70)</td>
</tr>
</tbody>
</table>
Conclusions

- Using the geometry of the RR chain to formulate the problem leads to a simple convenient set of equations.
- The design procedure for three positions for an RR chain yields a Bennett linkage.
- A Mathematica notebook with the complete synthesis procedure can be downloaded from: [http://www.eng.uci.edu/ mccarthy/Pages/ResProjects.html](http://www.eng.uci.edu/ mccarthy/Pages/ResProjects.html)
- The synthesis routine is to be used in a robot design environment for continuous tasks.