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DUAL QUATERNION SYNTHESIS OF A 2-TPR CONSTRAINED PARALLEL ROBOT

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Abstract: *This paper presents the dual quaternion synthesis methodology for constrained parallel robots. This methodology uses the dual quaternion form of the kinematics of supporting serial chains as design equations. Given a set of goal positions that define the desired workspace, we solve these design equations to determine the dimensions of the constrained parallel robot. The structure of these dual quaternion design equations allows a systematic elimination of the joint parameters. This methodology results in multiple solutions that are combined to form parallel robots. Here we formulate and solve the design equations for a 2-TPR parallel robot.*

1 Introduction

This paper presents a new formulation for the kinematic synthesis of constrained parallel robots. A constrained parallel robot is one in which each supporting chain imposes a kinematic constraint on the workpiece. These systems provide structural support in certain directions while allowing freedom of movement in others. Our synthesis methodology uses a set of goal positions that describe the workspace of the constrained parallel robot. The dual quaternion kinematic equations of the supporting chains are evaluated each of these goal positions to obtain the design equations, which are solved to obtain their physical dimensions.

This synthesis methodology is an extension of the kinematic synthesis of linkages (McCarthy (2000b)), which is based on finding the geometric constraints of the serial chain. The advantage of an approached based on the expression of the kinematic equations is that it can be applied systematically to serial chains with up to five degrees of freedom and joint axes. Multiple solutions obtained with this method can be combined to create a parallel robot.

The synthesis of parallel robotic systems has focussed on optimization strategies that allow the workpiece full mobility. Chedmail (1998) and Gosselin (1998) present optimization techniques for design serial and parallel robotic system, respectively, that provide desired properties of the workspace. Murray (2000) presents a similar methodology applied to planar platforms, and also Merlet (1997) presents an approach for six-degrees of freedom platforms that combines the geometric synthesis to enclose a given workspace and conditions to take into account joint limits and interferences. This paper focusses on a design methodology that results in a robotic system that guides a workpiece with less than full mobility.

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2 Literature Review

Spatial linkage synthesis uses the geometric properties of a serial chain to formulate algebraic equations that must be satisfied at each of a discrete set of positions in the workspace (Suh and Radcliffe (1978)). This yields algebraic equations that are solved to determine the dimensions of the chain. Also see McCarthy (2000). Examples of this are the synthesis of spatial RR chains (Tsai and Roth (1973), Perez and McCarthy (2000)), spatial CC chains (Chen and Roth (1969), Huang and Chang (2000)) and SS chains (Innocenti (1994), Liao and McCarthy (1998)). Larochelle (2000) uses planar quaternion optimization for the approximate synthesis of planar one degree-of-freedom linkages.

Recently, Mavroidis and Lee (2001) used the kinematics equations of the spatial RR and RRR robots to formulate their design equations. This approach introduces the joint parameters of the chain at each of the goal positions as additional variables in the design equations, see also Lee and Mavroidis (2002). The advantage is that it can be systematically applied to a broad range of robotic systems.

In this paper, we follow Mavroidis' basic ideas, however, we use successive screw displacements (Gupta (1986), Tsai (1999)) formulated in terms of dual quaternions to represent the kinematics equations of the robot. Dual quaternions were introduced to linkage analysis by Yang and Freudenstein (1964). They form an eight dimensional Clifford algebra that contains a subset, known as unit dual quaternions, which is isomorphic to the group of spatial displacements (McCarthy (1990)). Also see Angeles (1998).

There are two advantages in this formulation. The first is that successive screw displacements provide a convenient formulation for the kinematics equations in terms of the joint axes directly. Secondly, it reduces the number of equations obtained in each goal position from 12 to 8.

3 Supporting Chain Kinematics Equations

The kinematics equations of the robot equate the 4×4 homogeneous transformation $[D]$ between the end-effector and the base frame to the sequence of local coordinate transformations along the chain (Craig (1986)),

$$[D] = [G][Z(\theta_1, d_1)][X(\alpha_{12}, a_{12})][Z(\theta_2, d_2)] \dots \\ \dots [X(\alpha_{n-1,n}, a_{n-1,n})][Z(\theta_n, d_n)][H]. \quad (1)$$

The parameters (θ, d) define the movement at each joint and (α, a) are the length and twist of each link, collectively known as the Denavit-Hartenberg parameters. The transformation $[G]$ defines the position of the base of the chain relative to the fixed frame, and $[H]$ locates the tool relative to the last link frame.

3.1 Successive Screw Displacements

These kinematics equations can be transformed into successive screw displacements by choosing a reference position $[D_0]$. Let $[D_i]$ be the homogeneous matrix describing the transformation from the fixed frame to a moving frame F_i. We can compute $[D_{0i}] = [D_i][D_0]^{-1}$, that is,

$$[D_{0i}] = [D_i][D_0]^{-1} = \\ ([G][Z(\theta_{1i}, d_{1i})] \dots [Z(\theta_{ni}, d_{ni})][H]) \\ ([G][Z(\theta_{10}, d_{10})] \dots [Z(\theta_{n0}, d_{n0})][H])^{-1}. \quad (2)$$

This can be viewed as

$$[D_{0i}] = [T(\Delta\theta_1, \mathsf{S}_1)] \dots [T(\Delta\theta_n, \mathsf{S}_n)], \quad (3)$$

where

$$[T(\Delta\theta_1, \mathsf{S}_1)] = [G][Z(\theta_{1i}, d_{1i})][Z(\theta_{10}, d_{10})]^{-1}[G]^{-1}, \\ [T(\Delta\theta_2, \mathsf{S}_2)] = ([G][Z(\theta_{10}, d_{10})][X(\alpha_{12}, a_{12})][Z(\theta_{2i}, d_{2i})]) \\ ([G][Z(\theta_{10}, d_{10})][X(\alpha_{12}, a_{12})][Z(\theta_{20}, d_{20})])^{-1}, \\ \vdots \\ [T(\Delta\theta_n, \mathsf{S}_n)] = ([G][[Z(\theta_{10}, d_{10})] \dots) \\ [Z(\theta_n, d_n)][Z(\theta_{n0}, d_{n0})]^{-1}([G][[Z(\theta_{10}, d_{10})] \dots)^{-1}. \quad (4)$$

The displacements $[T(\Delta\theta_i, \mathsf{S}_i)]$ are the relative rotations about and translations along the joint axes S_i of the robot from the chosen reference configuration. Notice that by expressing them in this way, the initial transformation $[G]$ is absorbed in the first joint axis and the final transformation $[H]$ disappears from the expression.

3.2 Dual Quaternion Kinematics Equations

The workspace of the robot can also be expressed by using the Clifford algebra of the *dual quaternions*. A spatial displacement can be represented as a dual quaternion,

$$\hat{Q}(\hat{\theta}) = \sin(\frac{\hat{\theta}}{2})\mathsf{S} + \cos(\frac{\hat{\theta}}{2}), \quad (5)$$

where $\mathsf{S} = \mathbf{s} + \epsilon\mathbf{s}^0$, with $\epsilon^2 = 0$, is the screw axis of the transformation. The dual numbers $\cos(\frac{\hat{\theta}}{2}) = \cos\frac{\theta}{2} + \epsilon(-\frac{d}{2}\sin\frac{\theta}{2})$ and $\sin(\frac{\hat{\theta}}{2}) = \sin\frac{\theta}{2} + \epsilon(\frac{d}{2}\cos\frac{\theta}{2})$ contain the information about the rotation about and the displacement along the screw axis. The components of the dual quaternions can be easily computed from the homogeneous matrix transformation.

The spatial displacements can be represented as the set of points $\mathbf{Z} = (\mathbf{Z}, \mathbf{Z}^0)$ in $\mathbf{R}^8$ which are subject to two constraints: $\mathbf{Z}\cdot\mathbf{Z} = 1$ and $\mathbf{Z}\cdot\mathbf{Z}^0 = 0$. Then the workspace can be represented as lying on a six-dimensional submanifold of $\mathbf{R}^8$.

The dual quaternion form for the kinematics equations of the robot are obtained by transforming eq.(3) into

$$\hat{D}^i = \hat{S}_1(\Delta\hat{\theta}_1^i) \dots \hat{S}_n(\Delta\hat{\theta}_n^i), \quad (6)$$

where $\hat{D}^i$ is the dual quaternion for $[D_{0i}]$ and $\hat{S}_j$ is the dual quaternion for $[T(\Delta\theta_j, \mathsf{S}_j)]$.

This approach yields the kinematics equations as successive screw transformations from the reference position. It is a useful formulation from the synthesis point of view because the components of each axis appear explicitly in the base frame coordinates.

4 Constrained Parallel Robot Synthesis

The dual quaternion methodology for the synthesis of constrained serial chains yields multiple solutions. These solutions can be combined to form a parallel robot. It is also possible to design a different seral chain for the same set of goal positions and add this to the parallel robot. The operation of assembling the end-effectors of several serial chains ensures that the parallel robot will reach each of the goal positions of the supporting serial chains.

It is useful to note that the constraints on the workpiece of the combined system may not allow smooth movement through all of the goal positions. This is a performance issue that requires additional analysis. Other performance concerns are to accommodate joint limits as well as the potential for interference between links. Other performance factors can be included in the design process, such as dexterity, mechanical advantage, forces at the joints, the effect of tolerances, and positioning errors.

In this paper, we focus on the geometric problem of ensuring that the constraints imposed by each supporting chain are satisfied at each of the desired goal positions of the workpiece.

4.1 Design Equations for Supporting Serial Chains

Let $[T(\theta_1, \ldots, \theta_k)]$ be the kinematics equations of a serial robot, and let a discrete approximation of the desired workspace be given in the form of n goal transformations $[D_i], i = 0, \ldots, n-1$. The synthesis problem consists of solving the n matrix equations

$$[T(\theta_{1,i}, \ldots, \theta_{k,i})] = [D_i], \quad \imath = 0, \ldots, n-1. \tag{7}$$

We now transform these equations to successive screw displacements in dual quaternion form. Equating the $n-1$ goal positions $\hat{D}^i, i = 1, \ldots, n-1$ to the kinematics equations $\hat{Q}(\hat{\theta}_1, \ldots, \hat{\theta}_k)$, we obtain the $n-1$ equations

$$\hat{Q}_i(\hat{\theta}^i_1, \ldots, \hat{\theta}^i_k) = \hat{D}^i, \quad i = 1, \ldots, n-1 \tag{8}$$

For each of the $n-1$ positions we define eight component equations. However, due to the structure of the dual quaternions, only six of them are independent. For a unit dual quaternion, the 2-norm of the first vector is equal to one and the dot product of the first times the second vectors is equal to zero.

Assume for the moment that the robot chain can be represented by an equivalent series of j revolute joints. Each of these joints has an axis which is defined by six Plucker coordinates, which yields $6j$ unknowns. The j joint variables take different values at each of the $n-1$ positions, which add $j(n-1)$ unknowns. This yields $6j + j(n-1)$ unknowns.

Two constraint equations are associated with Plucker coordinates arise for each joint axis. For each of the $n-1$ goal positions we obtain eight equations, which can be reduced to six. Thus, we have $2j + 6(n-1)$ equations.

Equating the number of unknowns to the number equations, we obtain

$$6j + j(n-1) = 6(n-1) + 2j. \tag{9}$$

Solving for n

$$n = \frac{3j+6}{6-j}, \tag{10}$$

we have that $2R$, $3R$, $4R$ and $5R$ spatial chains require 3, 5, 9, 21 positions, respectively. However, we need to consider some limitations. In eq. (9) we equate dual quaternions component by component. As the rotations operate independently in spatial displacements, the number of spherical positions we can reach will be limited by this fact, while the number of spatial translations is computed in general. Hence, to compute complete spatial positions, first we need to check how these are limited by the maximum number of spherical positions we can reach. To separate rotations from translations, assume our robot consists of l rotational joints and k translational joints. We therefore need two equations; the first one equating rotational joint directions with rotation components of the dual quaternion,

$$3l + l(n_R - 1) = 3(n_R - 1) + l \tag{11}$$

and the second equating both rotational and translational joints to the whole quaternion,

$$6(l+k) + (l+k)(n-1) = 6(n-1) + 2l + k. \tag{12}$$

From the rotation equation,

$$n_R = \frac{3+l}{3-l}. \tag{13}$$

Notice that this coincides with the results for spherical robots: for one revolute joint we obtain finite number of solutions for two positions, this means we can reach one relative rotation. For two revolute joints we have finite number of solutions for $n_R = 5$, while for three we get infinity, which means that we can reach any orientation. The formula stops making sense after this. The maximum number of complete positions we can reach will be restricted by n_R, and if in the second formula we obtain more than that, the rest will be just translational components of dual quaternions in which rotations will have to be bounded to the given workspace.

Notice also that here we assume that the axes of the rotational and translational joints are not related, but it is easy to adapt the formula to particular cases in which the joints are constrained.

4.2 Solving the Design Equations

The design equations for constrained robots contain joint variables and the kinematic parameters defining the joint axes. Our goal is to eliminate the joint variables, if possible, and solve for the parameters of the axes, which define the physical dimensions of the robot.

In order to eliminate the joint parameters, we consider the equations for each position independently. We call this process "implicitization" of the parametric equations, see Cox (1998). The first step in this implicitization process uses the semi-direct product structure of the group of spatial displacements captured by the algebra of dual quaternions, which separates the composition of rotations in the real part from a combination of translations and rotations in the dual part. In the dual quaternion product the first four components are never mixed with the last four in any computation.

The four rotational components of the dual quaternion equation are parameterized only by the revolute joint variables,

$$\hat{Q}_{rot}(\theta_1, \ldots, \theta_k) = \begin{Bmatrix} q_x(\theta_1, \ldots, \theta_k) \\ q_y(\theta_1, \ldots, \theta_k) \\ q_z(\theta_1, \ldots, \theta_k) \\ q_w(\theta_1, \ldots, \theta_k) \end{Bmatrix} = \begin{Bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{Bmatrix} \tag{14}$$

This can always be transformed to a linear system that allows to solve for two of the revolute joint variables as a function of the joint axes and the rest of revolute variables,

$$[R(\theta_3,\ldots,\theta_k)]\begin{Bmatrix}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\\ \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\\ \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\\ \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\end{Bmatrix}=\begin{Bmatrix}p_x\\ p_y\\ p_z\\ p_w\end{Bmatrix} \tag{15}$$

where the matrix $[R(\theta_3,\ldots,\theta_k)]$ is invertible for non-degenerated cases. Degenerated cases are those in which the axes of the variables we are solving for are parallel, for instance solutions in which the serial chain is not spatial but planar. We can assume the matrix is compatible when the axes are a solution for the design problem. Also notice that the matrix in Eq. (15) is orthogonal for the cases in which the axes of the variables that we are solving for are perpendicular.

This allows us to eliminate linearly two of the rotational parameters in the form of a vector of sine and cosines. We can then substitute these expressions in the second four components of the dual quaternion,

$$\hat{Q}_{trans}(\theta_3,\ldots,\theta_k,d_1,\ldots,d_l)= \\ =\begin{Bmatrix}q_x^0(\theta_3,\ldots,\theta_k,d_1,\ldots,d_l)\\ q_y^0(\theta_3,\ldots,\theta_k,d_1,\ldots,d_l)\\ q_z^0(\theta_3,\ldots,\theta_k,d_1,\ldots,d_l)\\ q_w^0(\theta_3,\ldots,\theta_k,d_1,\ldots,d_l)\end{Bmatrix}=\begin{Bmatrix}p_x^0\\ p_y^0\\ p_z^0\\ p_w^0\end{Bmatrix}. \tag{16}$$

As a general rule, we can eliminate the last equation in Eq.(16), as we can see that in equating a robot $\hat{Q}$ to a goal dual quaternion $\hat{P}$, the equation $q_w^0 = p_w^0$ does not add anything to the set of solutions,

$$(q_x^0-p_x^0)p_x+(q_y^0-p_y^0)p_y+(q_z^0-p_z^0)p_z+(q_w^0-p_w^0)p_w- \\ (p_x-q_x)q_x^0-(p_y-q_y)q_y^0-(p_z-q_z)q_z^0- \\ (p_w-q_w)q_w^0+(q_x^0q_x+q_y^0q_y+q_z^0q_z+q_w^0q_w)=0. \tag{17}$$

To this set of equations we need to add any condition on the additional joint variables that is implicit in the solution for the rotations. The subsequent joint variables can be eliminated sequentially in a similar fashion, but the procedure is different from case to case. The parameters corresponding to revolute joints appear as quadratic sine and cosine functions, while the parameters corresponding to prismatic joints appear linearly.

To the final set of equations free of joint variables we need to add the Plucker conditions for each joint axis $\mathsf{S}_i = \mathbf{s}_i + \epsilon\mathbf{s}_i^0$; in fact, these equations are the ones that allow us to disregard two equations for each dual quaternion equality.

$$\mathbf{s}_i\cdot\mathbf{s}_i = 1, \quad i=1,\ldots,k+l \\ \mathbf{s}_i\cdot\mathbf{s}_i^0 = 0, \quad i=1,\ldots,k \tag{18}$$

In the example below the process is illustrated for a TPR chain.

5 Design of the 2-TPR Constrained Parallel Robot

The 2-TPR robot consists of an end-effector supported by two TPR serial chains. Each supporting TPR serial chain imposes two constraints on the end-effector, which means that the resulting system has two degrees of freedom.

The TPR serial chain is a four-degree of freedom robot. The base joint T consists of two revolute joints about perpendicular axes. This joint is also called U-joint for universal joint. The fixed axis G_1 allows rotation of angle θ_1 about it. Located at 90^o and intersecting G_1 is the second revolute axis, G_2, which allows rotation of angle θ_2. This is followed by a translation d along an arbitrary direction H and finally a rotation of angle ϕ about an arbitrary axis W, see Figure 1.

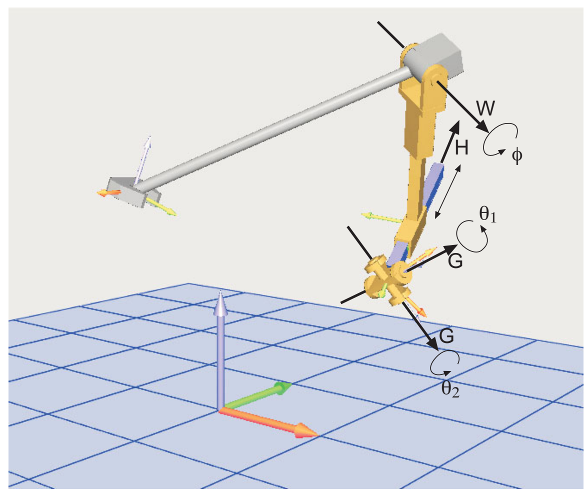

Figure 1: The spatial TPR robot

We call $\mathbf{c}$ to the intersection point of the two rotation axes G_1 and G_2. Notice that the location of the prismatic joint is immaterial and has been assigned in the drawing to the same intersection point.

The dual quaternion representation for the relative displacements of the chain is given by

$$\hat{Q}_{TPR} = \hat{G_1}(\theta_1,0)\hat{G_2}(\theta_2,0)\hat{H}(0,d)\hat{W}(\phi,0), \tag{19}$$

When applying the dual quaternion product we obtain the expression $\hat{Q}_{TPR} = Q^0 + \mathsf{Q}$, where the point is

$$Q^0 = c\frac{\theta_1}{2}c\frac{\theta_2}{2}c\frac{\phi}{2} - s\frac{\theta_1}{2}c\frac{\theta_2}{2}s\frac{\phi}{2}\mathsf{G}_1\cdot\mathsf{W} \\ - c\frac{\theta_1}{2}s\frac{\theta_2}{2}s\frac{\phi}{2}\mathsf{G}_2\cdot\mathsf{W} - s\frac{\theta_1}{2}s\frac{\theta_2}{2}s\frac{\phi}{2}(\mathsf{G}_1\times\mathsf{G}_2)\cdot\mathsf{W} \\ - \epsilon\frac{d}{2}\Big(s\frac{\theta_1}{2}c\frac{\theta_2}{2}c\frac{\phi}{2}\mathsf{G}_1\cdot\mathsf{H} + c\frac{\theta_1}{2}s\frac{\theta_2}{2}c\frac{\phi}{2}\mathsf{G}_2\cdot\mathsf{H} \\ + c\frac{\theta_1}{2}c\frac{\theta_2}{2}s\frac{\phi}{2}\mathsf{H}\cdot\mathsf{W} + s\frac{\theta_1}{2}s\frac{\theta_2}{2}c\frac{\phi}{2}(\mathsf{G}_1\times\mathsf{G}_2)\cdot\mathsf{H} \\ + s\frac{\theta_1}{2}c\frac{\theta_2}{2}s\frac{\phi}{2}(\mathsf{G}_1\times\mathsf{H})\cdot\mathsf{W} + c\frac{\theta_1}{2}s\frac{\theta_2}{2}s\frac{\phi}{2}(\mathsf{G}_2\times\mathsf{H})\cdot\mathsf{W} \\ + s\frac{\theta_1}{2}s\frac{\theta_2}{2}s\frac{\phi}{2}((\mathsf{G}_1\times\mathsf{G}_2)\times\mathsf{H})\cdot\mathsf{W}\Big), \tag{20}$$

and the dual vector,

$$\begin{aligned}
\mathsf{Q} &= s\frac{\theta_1}{2}c\frac{\theta_2}{2}c\frac{\phi}{2}\mathsf{G}_1 + c\frac{\theta_1}{2}s\frac{\theta_2}{2}c\frac{\phi}{2}\mathsf{G}_2\\
&+ c\frac{\theta_1}{2}c\frac{\theta_2}{2}s\frac{\phi}{2}\mathsf{W} + s\frac{\theta_1}{2}s\frac{\theta_2}{2}c\frac{\phi}{2}\mathsf{G}_1\times\mathsf{G}_2\\
&+ s\frac{\theta_1}{2}c\frac{\theta_2}{2}s\frac{\phi}{2}\mathsf{G}_1\times\mathsf{W} + c\frac{\theta_1}{2}s\frac{\theta_2}{2}s\frac{\phi}{2}\mathsf{G}_2\times\mathsf{W}\\
&+ s\frac{\theta_1}{2}s\frac{\theta_2}{2}s\frac{\phi}{2}(\mathsf{G}_1\times\mathsf{G}_2)\times\mathsf{W}\\
&+ \epsilon\frac{d}{2}\Big(c\frac{\theta_1}{2}c\frac{\theta_2}{2}c\frac{\phi}{2}\mathsf{H}\\
&+ s\frac{\theta_1}{2}c\frac{\theta_2}{2}s\frac{\phi}{2}((\mathsf{G}_1\times\mathsf{H})\times\mathsf{W} - (\mathsf{G}_1\cdot\mathsf{H})\mathsf{W})\\
&+ c\frac{\theta_1}{2}s\frac{\theta_2}{2}s\frac{\phi}{2}((\mathsf{G}_2\times\mathsf{H})\times\mathsf{W} - (\mathsf{G}_2\cdot\mathsf{H})\mathsf{W})\\
&+ s\frac{\theta_1}{2}c\frac{\theta_2}{2}c\frac{\phi}{2}\mathsf{G}_1\times\mathsf{H} + c\frac{\theta_1}{2}s\frac{\theta_2}{2}c\frac{\phi}{2}\mathsf{G}_2\times\mathsf{H}\\
&+ s\frac{\theta_1}{2}s\frac{\theta_2}{2}c\frac{\phi}{2}(\mathsf{G}_1\times\mathsf{G}_2)\times\mathsf{H} + c\frac{\theta_1}{2}c\frac{\theta_2}{2}s\frac{\phi}{2}\mathsf{H}\times\mathsf{W}\\
&+ s\frac{\theta_1}{2}s\frac{\theta_2}{2}s\frac{\phi}{2}(((\mathsf{G}_1\times\mathsf{G}_2)\times\mathsf{H})\times\mathsf{W} - ((\mathsf{G}_1\times\mathsf{G}_2)\cdot\mathsf{H})\mathsf{W})\Big).
\end{aligned} \tag{21}$$

The expansion of this equations componentwise leads to a set of equations in the components of the axes. The T-joint axis is formulated so that the coordinates of the intersection point $\mathbf{c}$ appear explicitly, as the point is also a design parameter,

$$\begin{aligned}
\mathsf{G1} &= (g_{1x}, g_{1y}, g_{1z}) + \epsilon((c_x, c_y, c_z)\times(g_{1x}, g_{1y}, g_{1z}))\\
\mathsf{G2} &= (g_{2x}, g_{2y}, g_{2z}) + \epsilon((c_x, c_y, c_z)\times(g_{2x}, g_{2y}, g_{2z})).
\end{aligned} \tag{22}$$

The moving prismatic axis has direction $\mathbf{h} = (h_x, h_y, h_z)$ and arbitrary location that will not appear in the design equations. The moving rotation axis is expressed in Plucker coordinates, $\mathsf{W} = (w_x, w_y, w_z) + \epsilon(w_x^0, w_y^0, w_z^0)$.

The number of positions needed to obtain finite number of solutions is calculated as explained in the previous section. As we have three rotational joints, the robot will be able to reach any orientation and the orientation does not limit the number of complete positions to reach. We have $18 + 4(n-1)$ unknowns, corresponding to the direction $\mathbf{G}_1$, the point $\mathbf{c}$, the direction $\mathbf{G}_2$, the direction $\mathbf{h}$ and the line W, plus the joint variables for the $n-1$ relative transformations. We have $6 + 6(n-1)$ equations, corresponding to the unit vector conditions for all directions, the perpendicularity condition between $\mathbf{G}_1$ and $\mathbf{G}_2$ and the moment condition for W, plus the six independent equations per goal dual quaternion . Therefore we need to define $n = 7$ positions.

5.1 The Design Equations

To create the design equations we equate the expanded eqs.(20, 21) to the goal dual quaternion $\hat{P}$, that is,

$$\hat{Q}_{TPR}(\theta_1^i, \theta_2^i, d^i, \phi^i) - \hat{P}^i = \vec{0}, \tag{23}$$

to obtain one of the sets of design equations. After equating for all the relative dual quaternion transformations, we obtain six sets of dual quaternion equations. However, to eliminate the joint parameters we work with only a generic set and apply the results to each relative position.

To eliminate the joint parameters we consider the separation between rotations and translations. It is easy to solve for two of the rotational joint parameters as shown in Eq.(15). Every direction will be reached by moving the rotation axes accordingly to the third rotation parameter as appears in the solution of the linear system,

$$[R(\phi)]\begin{Bmatrix}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\\ \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\\ \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\\ \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\end{Bmatrix} = \begin{Bmatrix}p_x\\ p_y\\ p_z\\ p_w\end{Bmatrix}, \tag{24}$$

with

$$[R(\phi)] = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4\\ s_1 & s_2 & s_3 & s_4\end{bmatrix}. \tag{25}$$

The column vectors in $[R(\phi)]$ are

$$\begin{aligned}
\mathbf{v}_1 &= \cos\frac{\phi}{2}\mathbf{g}_2 + \sin\frac{\phi}{2}\mathbf{g}_2\times\mathbf{w}\\
\mathbf{v}_2 &= \cos\frac{\phi}{2}\mathbf{g}_1 + \sin\frac{\phi}{2}\mathbf{g}_1\times\mathbf{w}\\
\mathbf{v}_3 &= \cos\frac{\phi}{2}\mathbf{g}_1\times\mathbf{g}_2 + \sin\frac{\phi}{2}((\mathbf{g}_1\times\mathbf{g}_2)\times\mathbf{w} - (\mathbf{g}_1\cdot\mathbf{g}_2)\mathbf{w})\\
\mathbf{v}_4 &= \sin\frac{\phi}{2}\mathbf{w}
\end{aligned} \tag{26}$$

and the last row is composed of the scalars

$$\begin{aligned}
s_1 &= -\sin\frac{\phi}{2}\mathbf{g}_2\cdot\mathbf{w}\\
s_2 &= -\sin\frac{\phi}{2}\mathbf{g}_1\cdot\mathbf{w}\\
s_3 &= -\cos\frac{\phi}{2}\mathbf{g}_1\cdot\mathbf{g}_2 - \sin\frac{\phi}{2}(\mathbf{g}_1\times\mathbf{g}_2)\cdot\mathbf{w}\\
s_4 &= \cos\frac{\phi}{2}.
\end{aligned} \tag{27}$$

The matrix $[R(\phi)]$ is an orthogonal matrix when solving for the joint variables θ_1, θ_2 corresponding to the T-joint. The solution for the angles is

$$\begin{Bmatrix}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\\ \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\\ \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\\ \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\end{Bmatrix} = [R(\phi)]^T\begin{Bmatrix}p_x\\ p_y\\ p_z\\ p_w\end{Bmatrix} \tag{28}$$

The solution always exists for directions $\mathbf{g}_1$, $\mathbf{g}_2$ and $\mathbf{w}$ and angles ϕ such that the system is solvable, which we can assume will be given by the solution of the design equations. In this case there is not planar degeneracy if we consider the constraint for

$\mathbf{g}_1$ and $\mathbf{g}_2$ to be at right angles. The angle ϕ is constrained by the relation among the four variables we are solving for,

$$\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cdot\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} = \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cdot\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}, \quad (29)$$

obtaining the condition for ϕ,

$$A_0\cos^2\frac{\phi}{2} + B_0\sin^2\frac{\phi}{2} + C_0\cos\frac{\phi}{2}\sin\frac{\phi}{2} = 0. \quad (30)$$

The solutions for the θ_1, θ_2 angles are substituted in the three first moment equations of the dual quaternion. We obtain three equations which are linear in the prismatic joint variable d and quadratic in the revolute joint variable ϕ, and that we denote by

$$\begin{aligned}&(A_{1i}d + A_{0i})\cos^2\frac{\phi}{2} + (B_{1i}d + B_{0i})\sin^2\frac{\phi}{2} + \\ &(C_{1i}d + C_{0i})\cos\frac{\phi}{2}\sin\frac{\phi}{2} + D_{0i} = 0, \quad i = 1, \dots, 3 \quad (31)\end{aligned}$$

To eliminate ϕ, we add the previously obtained angle condition, eq. (30), to create the homogeneous system

$$\begin{bmatrix} A_{11}d + A_{01} & B_{11}d + B_{01} & C_{11}d + C_{01} & D_{01} \\ \vdots & \vdots & \vdots & \vdots \\ A_{13}d + A_{03} & B_{13}d + B_{03} & C_{13}d + C_{03} & D_{03} \\ A_0 & B_0 & C_0 & 0 \end{bmatrix} \begin{Bmatrix} \cos^2\frac{\phi}{2} \\ \sin^2\frac{\phi}{2} \\ \cos\frac{\phi}{2}\sin\frac{\phi}{2} \\ 1 \end{Bmatrix} = \vec{0} \quad (32)$$

For the system to have solutions, the determinant must be equal to zero. The determinant is a quadratic equation in the prismatic joint variable d.

We can obtain the subspace of solutions from the matrix corresponding to the first three rows. By solving linearly in this system for the variables $\cos^2\frac{\phi}{2}$, $\sin^2\frac{\phi}{2}$ and $\cos\frac{\phi}{2}\sin\frac{\phi}{2}$, we obtain expressions as a function of the prismatic joint variable d. The relations between these three solutions,

$$\begin{aligned}&\cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2} = 1 \\ &(\cos^2\frac{\phi}{2})(\sin^2\frac{\phi}{2}) = (\cos\frac{\phi}{2}\sin\frac{\phi}{2})^2 \quad (33)\end{aligned}$$

lead to two more equations in d, the first one being a cubic and the second one a quartic in d. We obtain the system of three equations

$$\begin{aligned}K_{4i}d^4 + K_{3i}d^3 + K_{2i}d^2 + K_{1i}d + K_{0i} = 0,& \\ i = 1, \dots, 3.& \quad (34)\end{aligned}$$

Out of the system of three equations in d,

$$\begin{bmatrix} 0 & 0 & K_{21} & K_{11} & K_{01} \\ 0 & K_{32} & K_{22} & K_{12} & K_{02} \\ K_{43} & K_{33} & K_{23} & K_{13} & K_{03} \end{bmatrix} \begin{Bmatrix} d^4 \\ d^3 \\ d^2 \\ d \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (35)$$

we can eliminate the parameter d, for instance by direct substitution of the solution of d from the quadratic equation in the other two. We obtain two design equations per goal dual quaternion, which are free of joint variables and depend only on the coordinates of the joint axes. These 12 equations together with 6 constraints,

$$\begin{aligned}&g_{1x}^2 + g_{1y}^2 + g_{1z}^2 = 1, \\ &g_{2x}^2 + g_{2y}^2 + g_{2z}^2 = 1, \quad g_{1x}g_{2x} + g_{1y}g_{2y} + g_{1z}g_{2z} = 0 \\ &w_x^2 + w_y^2 + w_z^2 = 1, \quad w_x w_x^0 + w_y w_y^0 + w_z w_z^0 = 0 \\ &h_x^2 + h_y^2 + h_z^2 = 1 \quad (36)\end{aligned}$$

allows us to solve for the 18 unknowns corresponding to the four joint axes.

5.2 Assemble the 2-TPR Constrained Robot

From the design equations for the TPR serial chain, we will obtain a certain number of solutions. The exact number of complex solutions can be known if we are able to reduce the polynomial system of design equations to a triangular system with one polynomial being univariate. In the case of the serial TPR chain, the only possibility of creating a parallel robot is to form a 2-TPR robot by joining the end-link of two of the solutions. The 2-TPR robot has two degrees of freedom, and notice that the 3-TPR platform is a structure. The 2-TPR robot will reach the set of seven positions, but nothing ensures that the movement of the robot will be smooth or even possible. The simplest strategy to choose a good design is to create all possible combinations of two solutions and to analyze their movement through the goal positions.

6 Numerical Example

We present an example for the design of the 2-TPR parallel robot. To pick up to seven positions in space, we can either generate them individually or perform dual quaternion interpolation between an initial position, an intermediate position and a final position McCarthy and Ahlers (1999). Each TPR serial robot will exactly hit the seven positions in the trajectory. Another option is to set some of the parameters of the TPR chain to desired values and solve for a smaller number of positions.

In our example we solve for the seven positions for the first TPR serial chain, and for the second chain we set both the directions of the rotations of the T-joint $\mathbf{g}_1$ and $\mathbf{g}_2$ to a specified value, which is equivalent to fix the plane of the rotation, and we also impose the condition that the moving revolute joint axis W must be perpendicular to the prismatic joint direction $\mathbf{h}$. Using the

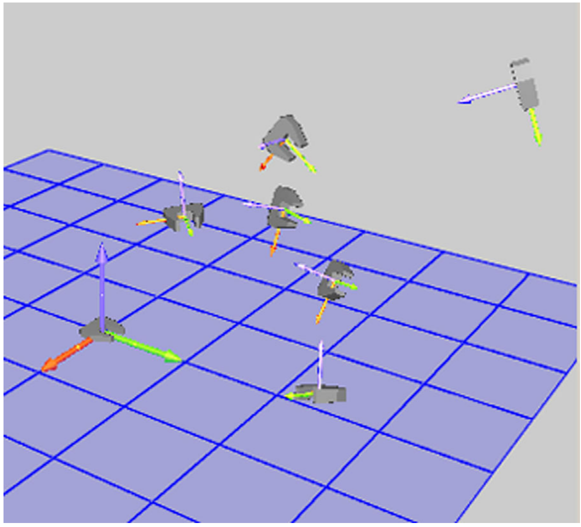

Figure 2: The seven initial positions

Table 1: THE GOAL POSITIONS

Pos.	*Axis*	*Rot.*	*Trans.*
1	$(1.0, 0.0, 0.0), (0.0, 0.0, 0.0)$	0^o	0
2	$(0.5, 0.8, -0.4), (-1.8, 0.8, -0.7)$	68.9^o	0.32
3	$(-0.2, 0.9, -0.3), (-1.7, -0.3, 0.2)$	92.7^o	0.71
4	$(0.0, 0.8, -0.5), (-2.2, 0.0, 0.2)$	156.5^o	1.39
5	$(0.3, 0.9, -0.3), (-1.6, 0.5, -0.1)$	79.0^o	0.31

counting formula, we see that we can solve for a finite number of solutions for $n = 5$ positions.

On Table 1 and Figure 2 we define and show the specified positions.

We solve numerically the design equations using a Newton-Raphson type of solver. The design equations are very sensitive to the initial conditions, and in this particular case we could not find any good solution for the second chain, and the numerical solver led to a local minimum that hits four of the five positions. In Table(2, 3) we can see the obtained solutions. Figure 3 shows the parallel 2-TPR robot while reaching positions 1, 2 and 5.

7 Conclusions

This paper introduces a new formulation for the kinematic synthesis of constrained parallel robots. While arbitrary serial chains can have up to six degrees of freedom, our focus is on chains with five or less degrees of freedom. These serial chains impose constraints on the workpiece of the parallel robot. These constraints can be used to provide structural support and enhance mechanical advantage.

The dual quaternion form of the kinematics equations of the supporting serial chain are evaluated at a set of goal positions to form design equations. These equations include both axis parameters that define the robot and joint parameters that define its configuration in a goal position. The structure of these equations provide a convenient strategy for the elimination of the joint parameters, which we demonstrate for the TPR serial chain. The parallel 2-TPR constrained robot is constructed by joining the end-links of two TPR solutions.

Table 2: THE JOINT AXES FOR FIRST TPR CHAIN

Joint Axis	*Direction*	*Moment*
G_1	$(0.52, 0.34, -0.78)$	$(-1.39, 1.06, -0.47)$
G_2	$(-0.41, 0.90, 0.11)$	$(-0.72, -0.48, 1.20)$
H	$(0.81, 0.46, 0.35)$	$(0.02, 0.54, -0.77)$
W	$(0.48, 0.86, -0.19)$	$(-1.83, 0.69, -1.49)$

Table 3: THE JOINT AXES FOR SECOND TPR CHAIN

Joint Axis	*Direction*	*Moment*
G_1	$(1.0, 0.0, 0.0)$	$(0.0, 0.98, -2.15)$
G_2	$(0.0, 1.0, 0.0)$	$(-0.98, 0.0, 1.0)$
H	$(-0.68, -0.33, 0.66)$	$(1.74, -1.33, 1.12)$
W	$(0.49, -0.87, 0.08)$	$(1.29, 0.58, -1.75)$

Our goal is to expand this approach to a systematic design procedure for a wide range of constrained parallel robotic systems. So far, we have results for the RR, RP, RPR, RRR, CC, and TS serial chains. In addition, we look forward to formulating the design equations for the TPT serial chain. The main challenge is the increasing complexity of the joint parameters in the design equations. We also look forward to incorporating performance measures such as speed ratios and mechanical advantage in order to rate resulting designs.

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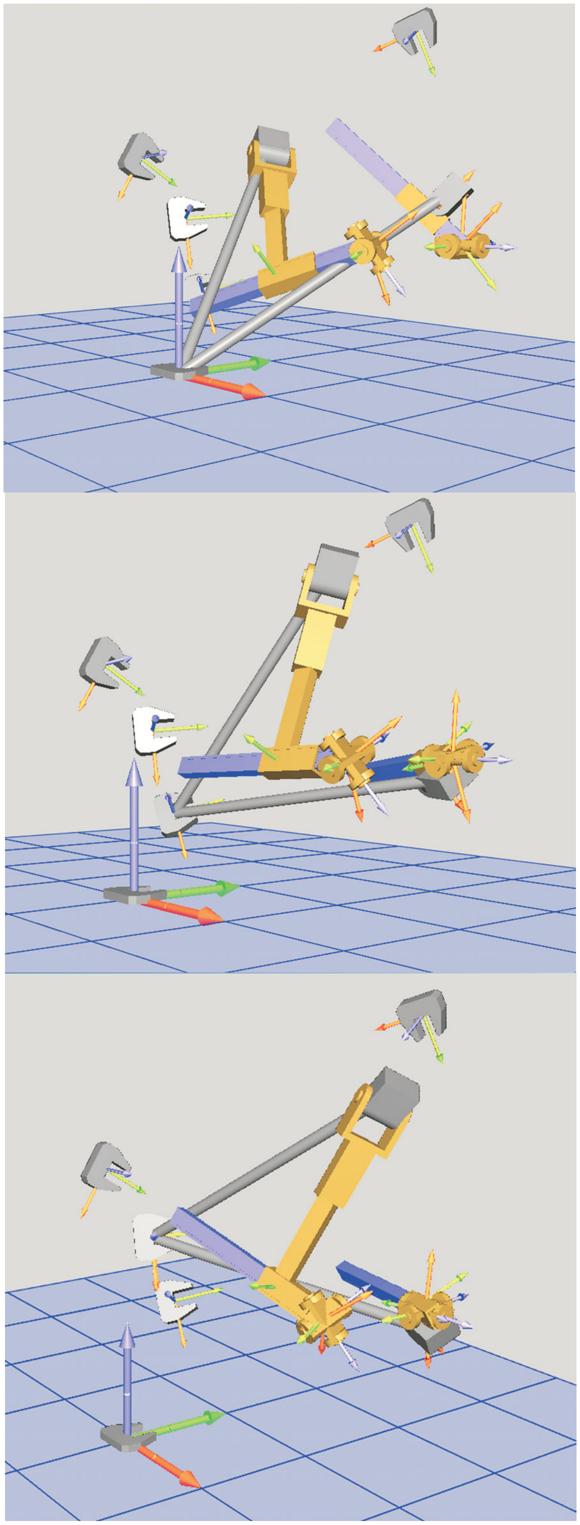

Figure 3: The 2-TPR robot reaching positions 1, 2 and 5