Slowing-down distances and times of 0.1- to 14-MeV neutrons in hydrogenous materials*

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The slowing-down distances and times of fast neutrons are calculated by extending the early work of Fermi to the case in which the neutron mean free path is energy dependent. This calculation is essentially a one-particle Monte Carlo treatment in which each path length is a mean free path and the neutron is always scattered through a scattering angle \( \theta \), such that \( \cos \theta = \langle \cos \theta \rangle \), the average of \( \cos \theta \) in the laboratory system. The average results derived by this analytic method are shown to be in reasonable agreement with a detailed Monte Carlo computer calculation of the moderation of a 1-MeV neutron down to 40 eV in pure hydrogen. These calculations illustrate the Monte Carlo method and the broad spread in time and energy of the neutron flux resulting from an instantaneous point source in hydrogen. The results also graphically illustrate the greater penetrating distances of 14-MeV neutrons compared to 1-MeV neutrons in hydrogenous materials. Pedagogic, computer-oriented uses of this material are suggested.

I. INTRODUCTION

A fundamental question in the design of neutron shielding is how far from a point source does a neutron of initial energy \( E_1 \) random-walk before its energy is moderated to some lower energy \( E_f \). The classic discussion of this problem was developed by Fermi.\(^1\) His treatment, however, was restricted to the case in which the neutron mean free path \( \lambda \) is constant or slowly varying with energy. As is well known, hydrogen is the most efficient neutron moderator, and the neutron scattering cross section of \(^1\)H is neither constant nor slowly varying in the range \( 0.1 \leq E \leq 14 \) MeV.\(^2\) Hence, to determine how far a neutron wanders initially in this energy range before it reaches a given lower energy, the Fermi treatment is generalized to the case in which \( \lambda \) is energy dependent. This method is applicable to any nucleus for which the dominant neutron scattering mechanism is elastic and for which the sequence of mean free paths describing the moderation process \( E_1 \rightarrow E_f \) is known.

II. DERIVATION

Consider a neutron of initial energy \( E_1 \) starting at the origin \( O \) in Fig. 1. The neutron will traverse a sequence of linear displacements \( r_j \) each terminated by a scattering event that sends the neutron off in a new direction. After the \( n \)th path the net displacement \( \mathbf{R} \) of the neutron from the origin is given by

\[
\mathbf{R} = \sum_{j=1}^{n} \mathbf{r}_j,
\]

and the magnitude squared of the net displacement is

\[
R^2 = \sum_{j=1}^{n} r_j \cdot \sum_{k=1}^{n} r_k.
\]

Equation (2) can be rewritten as

\[
R^2 = \sum_{j=1}^{n} r_j^2 + 2 \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} r_j r_k \cos \theta_{jk},
\]

where \( \theta_{jk} \) is the angle between the \( j \)th and \( k \)th paths. Averaging over the spatial variables and using the fact that the average of a product of independent variables equals the product of the averages, we obtain

\[
\langle R^2 \rangle = \sum_{j=1}^{n} \langle r_j^2 \rangle + 2 \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \langle r_j \rangle \langle r_k \rangle \langle \cos \theta_{jk} \rangle.
\]

Neutrons scatter with a radial distribution function \( e^{-r/\lambda} \), where \( \lambda \) is the total mean free path. Hence,

\[
\langle r_j \rangle = \lambda_j,
\]

and

\[
\langle r_j^2 \rangle = 2 \lambda_j^2.
\]

Fermi has shown\(^1\) that

\[
\langle \cos \theta_{jk} \rangle = \langle \cos \theta \rangle^{k-j} = \mu^{k-j}.
\]

If Eqs. (5) and (6) are substituted into Eq. (4),

\[
\langle R^2 \rangle = 2 \left[ \sum_{j=1}^{n} \lambda_j^2 + \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \lambda_j \lambda_k \mu^{k-j} \right].
\]

If \( \lambda \) is constant, then

\[
\langle R^2 \rangle = 2 \lambda^2 \left[ n + \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \mu^{k-j} \right].
\]

Fig. 1. Illustration of the net displacement \( \mathbf{R} \) of a neutron which has traversed a sequence of displacements \( \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n \), where each displacement is terminated by a scattering event.
Table 1. The sequence of neutron energy, average energy deposited locally per collision, total mean free path, and time interval to traverse the mean free path over the range 14 MeV down to thermal in hydrogen at density $8 \times 10^{22}$ cm$^{-3}$.

<table>
<thead>
<tr>
<th>$E$ (MeV)</th>
<th>$\Delta E$ (MeV)</th>
<th>$\lambda_T$ (cm)</th>
<th>$\Delta t$ (sec)</th>
</tr>
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<tr>
<td>14.1</td>
<td>7.22</td>
<td>18.4</td>
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<td>6.74</td>
<td>3.50</td>
<td>9.77</td>
<td>$2.73 \times 10^{-8}$</td>
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<td>5.90</td>
<td>$2.36 \times 10^{-8}$</td>
</tr>
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<td>3.78</td>
<td>$2.22 \times 10^{-8}$</td>
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<td>1.71</td>
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<td>1.16</td>
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<td>0.85</td>
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<td>0.62</td>
<td>$1.04 \times 10^{-8}$</td>
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<tr>
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<td>0.62</td>
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<tr>
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<td>0.61</td>
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<td>0.00</td>
<td>0.60</td>
<td>$2.04 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

The second term in brackets in Eq. (8) can be evaluated by noting that it is a double geometric sum. Hence,

$$\sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \mu^{j-k} = \frac{(n-1)\mu}{1-\mu} \left( \frac{\mu}{1-\mu} \right) \left( 1 - \mu^{n-1} \right). \tag{9}$$

If Eq. (9) is substituted into Eq. (8) and simplified,

$$\langle R^2 \rangle = \frac{2n\lambda^2}{1-\mu} \left[ 1 - \frac{\mu}{n} \left( \frac{1 - \mu^n}{1 - \mu} \right) \right]. \tag{10}$$

For the case $n = 1$, Eq. (10) reduces to Eq. (5b), as required. For large $n$ or small $\mu$, or both,

$$\langle R^2 \rangle \approx \frac{2n\lambda^2}{1-\mu}. \tag{11}$$

Equation (11) is the result obtained by Fermi. \(^1\)

III. NUMERICAL EXAMPLE: HYDROGEN

If the mean free path is not a constant with respect to energy, then the mean-square distance required to moderate a neutron from energy $E_i$ to $E_f$ can be calculated from Eq. (7). For the sake of discussion, consider an atomic density of hydrogen equal to the hydrogen density in CH$_2$, namely, $N = 8 \times 10^{22}$ cm$^{-3}$. This density is about twice that of liquid hydrogen. CH$_2$ is a material often used for neutron moderation. Table 1 shows the sequence of energy, average energy deposited locally per collision, and total mean free path for a neutron with initial energy $E_i = 14$ MeV and final energy $E_f = 3.27 \times 10^{-8}$ MeV, which is close to the room-temperature thermal energy of $2.5 \times 10^{-8}$ MeV. The data were taken from Ref. 2. Energy $E_{i+j+1}$ was obtained by subtracting from $E_{i}$ the average energy deposited locally in the $j$th collision and rounding off to the nearest of the 163 energy-group boundaries spanning the total energy range under consideration. The total mean free path was calculated from the relation

$$\lambda_T = (N\sigma_T)^{-1}, \tag{12}$$

where $\sigma_T$ is the total cross section. The average cosine of the scattering angle in the laboratory system is given by

$$\mu = 2/3A, \tag{13}$$

where $A$ is the atomic weight in neutron mass units and neutron scattering is isotropic in the center-of-mass system. Neutron scattering by $^1$H is isotropic in the center-of-mass system except for small anisotropy in the 14-MeV energy range, which means an increase of only a few percent in the value of $\mu = 7/3$ from Eq. (13). Hence, $\mu$ was assumed equal to $7/3$ at all energies. The average local energy deposits listed in Ref. 2 are for neutrons scattered with $\cos \theta = \mu$, which is precisely what is required for this calculation. The root-mean-squared slowing-down distance, $R_{rms} = \langle R^2 \rangle^{1/2}$, was calculated from Eq. (7) for all values of $E_i$ above 0.1 MeV in Table 1 and for three final energies, $E_f$: thermal, 40 eV, and 4 keV, respectively. The results are shown in Fig. 2.

The most graphic result of this calculation is the illustration of the penetrating power of 14-MeV neutrons. For example, it takes about 10 cm of CH$_2$ to thermalize a 1-MeV neutron, while 41 cm are required for a 14-MeV neutron. This result is due to the decrease in the $^1$H cross section as energy increases from 1 to 14 MeV. The slowing-down distance of a 14-MeV neutron is, in fact, almost independent of the final energy. It takes rms distances of 40.0 and 36.8 cm to slow down 14-MeV neutrons to 0.13 and 1.5 MeV, respectively.

The slowing-down times can be calculated by summing the time, $\Delta t$, spent at each energy. In particular, the average time, $\bar{t}$, required to moderate from energy $E_i$ to $E_{i+1}$ is given by

$$\bar{t} = \sum_{j=1}^{n} \Delta t_j = \sum_{j=1}^{n} \frac{\lambda_j}{\nu_j} \tag{14}$$

![Fig. 2. Average distance $R_{rms}$ for a neutron with initial energy $E_i$ to slow down to a final energy $E_f$ as a function of $E_i$ for several fixed values of $E_f$. The curves were calculated with Eq. (7) using the mean free paths in Table 1.](https://example.com/fig2.jpg)
where \( v_j \) is the neutron velocity at energy \( E_j \). The values of \( \Delta t_j \) are also given in Table I. Figure 3 is a plot of the average integrated time \( \bar{t} \) necessary to moderate a neutron from 14 MeV to a final energy \( E_f \). The figure shows, for example, that it takes about \( 1.6 \times 10^{-7} \) and \( 4.7 \times 10^{-6} \) sec to moderate a 14-MeV neutron to 40 eV and to room temperature, respectively.

Finally, it should be noted that both \( R_{rms} \) and \( \bar{t} \) depend on the \( \lambda_j \) to the first power. Since the \( \lambda_j \)'s are proportional to \( N^{-1} \), these results all scale inversely with the atom density of hydrogen.

**IV. COMPARISON WITH A DETAILED MONTE CARLO CALCULATION**

Section II is essentially a one-particle Monte Carlo calculation in which that particle always travels the average distance \( \lambda_j \) at \( E_j \) and is always scattered through \( \mu \), the average cosine of the scattering angle in the laboratory system. In reality the path lengths and scattering angles will cover a broad distribution. Hence, the plots in Figs. 2 and 3 give only average values. To check these results and to investigate the distribution in energy and time of the neutron flux, a detailed Monte Carlo calculation was performed.

The sample problem consists of a point source of 1-MeV neutrons emitted with a delta-function time distribution from the center of a sphere of hydrogen with atomic density \( N = 8 \times 10^{22} \) cm\(^{-3}\) and an outer radius of 50 cm. Since a 1-MeV neutron is thermalized in about 10 cm (Fig. 2), the distance between 10 and 50 cm corresponds to about 10 mean free paths. Hence, the medium is infinite for all practical purposes. Figure 2 shows that at a distance of 9 cm from the source the energy spectrum should be peaked in the vicinity of 40 eV; Fig. 3 shows that this peak in the 40-eV neutron flux should occur in time at about \( 1.6 \times 10^{-7} \) sec after the neutron is emitted. The TARTNP Monte Carlo neutron-transport code\(^5\) was used to check these results. The path lengths are chosen randomly from a distribution weighted by the total cross section, and the scattering angles are picked randomly from a distribution weighted by the known angular dependence of the cross section. The local energy deposited in a collision is then calculated on the basis of the conservation of energy and momentum. Two million Monte Carlo particles were followed using 78 energy groups and equal weighting for all particles. The flux at a radius of 9 cm was obtained by counting the number of particles transported into a zone 0.01 cm thick at this radius. The time rate \( \dot{N} \) of neutrons into this thin zone in the energy range \( 33 \leq E \leq 47 \) eV is shown in Fig. 4. The bars parallel to the abscissa represent the time bins employed, while the bars parallel to the ordinate represent the variance in the mean of 25 samples of 80,000 particles each. Figure 3 shows that the 40-eV flux should peak at about \( 1.6 \times 10^{-7} \) sec, and Fig. 4 shows this to be the case (it takes only about \( 0.1 \times 10^{-7} \) sec to moderate from 14 to 1 MeV). The width of the pulse is quite wide compared to the time of the pulse, having a full width at half-maximum (FWHM) of \( 2.65 \times 10^{-7} \) sec. The integral under the curve of Fig. 4 indicates that \( 8.5 \times 10^{-3} \) of the source neutrons pass through 9 cm in the range \( 33 \leq E \leq 47 \) eV.

The energy spectrum \( \eta \) at 9 cm integrated over the time interval \( (1.3-1.9) \times 10^{-7} \) sec (at the peak in Fig. 4) is shown in Fig. 5. The bars of the histogram parallel to the abscissa represent the energy bins used in this range. Representative variances of the mean flux at a given energy are indicated by the error bars. The spectrum is peaked near 10 eV rather than 40 eV as indicated by Fig. 2. However, Fig. 2 also shows that there is little sensitivity of the final energy \( E_f \) to \( R_{rms} \). In particular \( E_f \) changes by a factor of 2 for a change in \( R_{rms} \) of only 0.1 cm out of 9.0 cm. Thus, the comparison of a predicted peak at 40 eV from Eq. (7) as compared to the result of 10 eV from the detailed calculation is actually quite good. The peak would be expected to shift to 40 eV if...
the flux were calculated at 8.8 cm. As seen in Fig. 5 the energy spectrum is very broad, having a FWHM of about two orders of magnitude in energy.

V. PEDAGOGIC USES OF THIS MATERIAL

In addition to providing a simple illustration of the Monte Carlo method, this material provides a suitable problem for incorporating the use of the computer into physics education. For example, the student could be asked to write a computer program for the evaluation of Eq. (7) with the data in Table I. His results could easily be checked against Fig. 2. Next the student could extend his calculation in several ways of varying difficulty:

1. Repeat the calculation for other nuclei and compare results. This comparison would have to be over a relatively smaller energy interval because of the large number of collisions required for other nuclei.

2. Fit cross section data for $^1$H, say, to some appropriate functional dependence on energy and then incorporate into his program the calculation of the sequence of mean free paths and energies, rather than use Table I.

3. Incorporate the random sampling techniques discussed by Carter and Cashwell\(^5\) to relax the constraints that the cosine of the scattering angle is always $\mu$ and each path length is always a mean free path. By following just a few particles this way the student would gain an appreciation for the wide distributions in time and energy in Figs. 4 and 5.

VI. CONCLUSION

The average results derived from Eqs. (7) and (14) are in reasonable agreement with a detailed Monte Carlo calculation. These calculations illustrate the Monte Carlo method for neutron transport and illustrate the broad spread in time and energy of the neutron flux resulting from an instantaneous point source in hydrogen. Finally, Fig. 2 illustrates the greater penetrating distances of 14-MeV neutrons compared to 1-MeV neutrons in hydrogenous materials. Implicit in this difference is the greater difficulty in shielding a controlled fusion reactor compared to a fission reactor.

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\(^1\)E. Fermi, Nuclear Physics, with notes compiled by J. Orcar, A. H. Rosenfeld, and R. A. Schluter (University of Chicago Press, IL, 1950), pp. 181-186.


\(^3\)\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.


\(^5\)The TARTNP code uses physics and Monte Carlo techniques very similar to those described in L. L. Carter and E. D. Cashwell, Particle-Transport Simulation with the Monte Carlo Method (U.S. Energy Research and Development Administration, Oak Ridge, 1975). The use of this code is described in J. Kimlinger and E. Plechaty, Lawrence Livermore Laboratory Report UCID-17026 (1976).

\(^6\)The energy group structure was collapsed somewhat in the energy range in which the spectrum was to be calculated in order to improve the statistics. No energy groups above 1 MeV were used. These two things account for the difference in the number (162) of energy groups used to generate Table I and the number (78) used to do the detailed calculation.